



Supplementary Material: Computational Serendipity and Tensor Product Finite Element Differential Forms

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Lists of $\mathcal{S}_r \Lambda^k$ and $\mathcal{S}_r^- \Lambda^k$ basis functions

Below are computational basis functions for the serendipity space $\mathcal{S}_r \Lambda^k(\square_3)$ and the trimmed serendipity space $\mathcal{S}_r^- \Lambda^k(\square_3)$ for $r = 1$ to 3 , $k = 0$ to 2 , and $n = 3$. To the best of our knowledge, for $k = 1$ and $k = 2$, these bases have not appeared in the literature previously. All bases were generated using the *SageMath* code included with the supplementary materials in the file `construct-tools-n3.sage` and verified using the Basis Verification Algorithm described in the paper. The standard conversion between 1-forms and vectors in \mathbb{R}^3 is given by

$$a \, dx + b \, dy + c \, dz \longleftrightarrow [a \quad b \quad c]^T.$$

The standard conversion between 2-forms and vectors in \mathbb{R}^3 is given by

$$q \, dydz + r \, dx dz + s \, dx dy \longleftrightarrow [q \quad r \quad s]^T.$$

$$\mathcal{S}_1 \Lambda^0(\square_3), \mathcal{S}_1^- \Lambda^0(\square_3) \quad \begin{array}{l} \overline{(x+1)(y+1)(z+1)} \\ (x+1)(y+1)(z-1) \\ (x+1)(y-1)(z+1) \\ (x+1)(y-1)(z-1) \\ (x-1)(y+1)(z+1) \\ (x-1)(y+1)(z-1) \\ (x-1)(y-1)(z+1) \\ (x-1)(y-1)(z-1) \end{array}$$

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$$\begin{array}{l}
 \overline{(x+1)(y+1)(z+1)} \\
 (x+1)(y+1)(z-1) \\
 (x+1)(y-1)(z+1) \\
 (x+1)(y-1)(z-1) \\
 (x-1)(y+1)(z+1) \\
 (x-1)(y+1)(z-1) \\
 (x-1)(y-1)(z+1) \\
 (x-1)(y-1)(z-1) \\
 (x^2-1)(y+1)(z+1) \\
 (x^2-1)(y+1)(z-1) \\
 \mathcal{S}_2\Lambda^0(\square_3), \mathcal{S}_2^-\Lambda^0(\square_3) \quad (x^2-1)(y-1)(z+1) \\
 (x^2-1)(y-1)(z-1) \\
 (y^2-1)(x+1)(z+1) \\
 (y^2-1)(x-1)(z+1) \\
 (y^2-1)(x+1)(z-1) \\
 (y^2-1)(x-1)(z-1) \\
 (z^2-1)(x+1)(y+1) \\
 (z^2-1)(x+1)(y-1) \\
 (z^2-1)(x-1)(y+1) \\
 (z^2-1)(x-1)(y-1)
 \end{array}$$

$$\begin{array}{l}
 \overline{(x+1)(y+1)(z+1)} \\
 (x+1)(y+1)(z-1) \\
 (x+1)(y-1)(z+1) \\
 (x+1)(y-1)(z-1) \\
 (x-1)(y+1)(z+1) \\
 (x-1)(y+1)(z-1) \\
 (x-1)(y-1)(z+1) \\
 (x-1)(y-1)(z-1) \\
 (x^2-1)(y+1)(z+1) \\
 (x^2-1)(y+1)(z-1) \\
 (x^2-1)(y-1)(z+1) \\
 (x^2-1)(y-1)(z-1) \\
 (y^2-1)(x+1)(z+1) \\
 (y^2-1)(x-1)(z+1) \\
 (y^2-1)(x+1)(z-1) \\
 (y^2-1)(x-1)(z-1) \\
 \mathcal{S}_3\Lambda^0(\square_3), \mathcal{S}_3^-\Lambda^0(\square_3) \quad (z^2-1)(x+1)(y+1) \\
 (z^2-1)(x+1)(y-1) \\
 (z^2-1)(x-1)(y+1) \\
 (z^2-1)(x-1)(y-1) \\
 (x^2-1)x(y+1)(z+1) \\
 (x^2-1)x(y+1)(z-1) \\
 (x^2-1)x(y-1)(z+1) \\
 (x^2-1)x(y-1)(z-1) \\
 (y^2-1)(x+1)y(z+1) \\
 (y^2-1)(x-1)y(z+1) \\
 (y^2-1)(x+1)y(z-1) \\
 (y^2-1)(x-1)y(z-1) \\
 (z^2-1)(x+1)(y+1)z \\
 (z^2-1)(x+1)(y-1)z \\
 (z^2-1)(x-1)(y+1)z \\
 (z^2-1)(x-1)(y-1)z
 \end{array}$$

COMPUTATIONAL FINITE ELEMENT DIFFERENTIAL FORMS (SUPPLEMENT)

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$
$\mathcal{S}_1\Lambda^1(\square_3)$	$2x(y+1)(z+1)$	$(x^2-1)(z+1)$	$(x^2-1)(y+1)$
	$2x(y+1)(z-1)$	$(x^2-1)(z-1)$	$(x^2-1)(y+1)$
	$2x(y-1)(z+1)$	$(x^2-1)(z+1)$	$(x^2-1)(y-1)$
	$2x(y-1)(z-1)$	$(x^2-1)(z-1)$	$(x^2-1)(y-1)$
	$(y^2-1)(z+1)$	$2(x+1)y(z+1)$	$(y^2-1)(x+1)$
	$(y^2-1)(z+1)$	$2(x-1)y(z+1)$	$(y^2-1)(x-1)$
	$(y^2-1)(z-1)$	$2(x+1)y(z-1)$	$(y^2-1)(x+1)$
	$(y^2-1)(z-1)$	$2(x-1)y(z-1)$	$(y^2-1)(x-1)$
	$(z^2-1)(y+1)$	$(z^2-1)(x+1)$	$2(x+1)(y+1)z$
	$(z^2-1)(y-1)$	$(z^2-1)(x+1)$	$2(x+1)(y-1)z$
	$(z^2-1)(y+1)$	$(z^2-1)(x-1)$	$2(x-1)(y+1)z$
	$(z^2-1)(y-1)$	$(z^2-1)(x-1)$	$2(x-1)(y-1)z$

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$
	$x(y+1)(z+1)$	0	0
	$x(y+1)(z-1)$	0	0
	$x(y-1)(z+1)$	0	0
	$x(y-1)(z-1)$	0	0
	0	$(x+1)y(z+1)$	0
	0	$(x-1)y(z+1)$	0
	0	$(x+1)y(z-1)$	0
	0	$(x-1)y(z-1)$	0
	0	0	$(x+1)(y+1)z$
	0	0	$(x+1)(y-1)z$
	0	0	$(x-1)(y+1)z$
	0	0	$(x-1)(y-1)z$
$\mathcal{S}_2\Lambda^1(\square_3)$	$3x^2(y+1)(z+1)$	$(x^2-1)x(z+1)$	$(x^2-1)x(y+1)$
	$3x^2(y+1)(z-1)$	$(x^2-1)x(z-1)$	$(x^2-1)x(y+1)$
	$3x^2(y-1)(z+1)$	$(x^2-1)x(z+1)$	$(x^2-1)x(y-1)$
	$3x^2(y-1)(z-1)$	$(x^2-1)x(z-1)$	$(x^2-1)x(y-1)$
	$(y^2-1)y(z+1)$	$3(x+1)y^2(z+1)$	$(y^2-1)(x+1)y$
	$(y^2-1)y(z+1)$	$3(x-1)y^2(z+1)$	$(y^2-1)(x-1)y$
	$(y^2-1)y(z-1)$	$3(x+1)y^2(z-1)$	$(y^2-1)(x+1)y$
	$(y^2-1)y(z-1)$	$3(x-1)y^2(z-1)$	$(y^2-1)(x-1)y$
	$(z^2-1)(y+1)z$	$(z^2-1)(x+1)z$	$3(x+1)(y+1)z^2$
	$(z^2-1)(y-1)z$	$(z^2-1)(x+1)z$	$3(x+1)(y-1)z^2$
	$(z^2-1)(y+1)z$	$(z^2-1)(x-1)z$	$3(x-1)(y+1)z^2$
	$(z^2-1)(y-1)z$	$(z^2-1)(x-1)z$	$3(x-1)(y-1)z^2$
	$(y^2-1)(z+1)$	0	0
	$(y^2-1)(z-1)$	0	0
	$(z^2-1)(y+1)$	0	0
	$(z^2-1)(y-1)$	0	0
	0	$(x^2-1)(z+1)$	0
	0	$(x^2-1)(z-1)$	0
	0	$(z^2-1)(x+1)$	0
	0	$(z^2-1)(x-1)$	0
	0	0	$(x^2-1)(y+1)$
	0	0	$(x^2-1)(y-1)$
	0	0	$(y^2-1)(x+1)$
	0	0	$(y^2-1)(x-1)$

A. GILLETTE, T. KLOEFKORN, *et al.*

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$
	$x(y+1)(z+1)$	0	0
	$x(y+1)(z-1)$	0	0
	$x(y-1)(z+1)$	0	0
	$x(y-1)(z-1)$	0	0
	0	$(x+1)y(z+1)$	0
	0	$(x-1)y(z+1)$	0
	0	$(x+1)y(z-1)$	0
	0	$(x-1)y(z-1)$	0
	0	0	$(x+1)(y+1)z$
	0	0	$(x+1)(y-1)z$
	0	0	$(x-1)(y+1)z$
	0	0	$(x-1)(y-1)z$
	$x^2(y+1)(z+1)$	0	0
	$x^2(y+1)(z-1)$	0	0
	$x^2(y-1)(z+1)$	0	0
	$x^2(y-1)(z-1)$	0	0
	0	$(x+1)y^2(z+1)$	0
	0	$(x-1)y^2(z+1)$	0
	0	$(x+1)y^2(z-1)$	0
	0	$(x-1)y^2(z-1)$	0
	0	0	$(x+1)(y+1)z^2$
	0	0	$(x+1)(y-1)z^2$
	0	0	$(x-1)(y+1)z^2$
	0	0	$(x-1)(y-1)z^2$
	$4x^3(y+1)(z+1)$	$(x^2-1)x^2(z+1)$	$(x^2-1)x^2(y+1)$
	$4x^3(y+1)(z-1)$	$(x^2-1)x^2(z-1)$	$(x^2-1)x^2(y-1)$
	$4x^3(y-1)(z+1)$	$(x^2-1)x^2(z+1)$	$(x^2-1)x^2(y-1)$
	$4x^3(y-1)(z-1)$	$(x^2-1)x^2(z-1)$	$(x^2-1)x^2(y-1)$
	$(y^2-1)y^2(z+1)$	$4(x+1)y^3(z+1)$	$(y^2-1)(x+1)y^2$
	$(y^2-1)y^2(z-1)$	$4(x-1)y^3(z-1)$	$(y^2-1)(x-1)y^2$
	$(y^2-1)y^2(z+1)$	$4(x+1)y^3(z-1)$	$(y^2-1)(x+1)y^2$
	$(y^2-1)y^2(z-1)$	$4(x-1)y^3(z-1)$	$(y^2-1)(x-1)y^2$
	$(z^2-1)(y+1)z^2$	$(z^2-1)(x+1)z^2$	$4(x+1)(y+1)z^3$
	$(z^2-1)(y-1)z^2$	$(z^2-1)(x+1)z^2$	$4(x+1)(y-1)z^3$
	$(z^2-1)(y+1)z^2$	$(z^2-1)(x-1)z^2$	$4(x-1)(y+1)z^3$
	$(z^2-1)(y-1)z^2$	$(z^2-1)(x-1)z^2$	$4(x-1)(y-1)z^3$
	$(y^2-1)(z+1)$	0	0
	$(y^2-1)(z-1)$	0	0
	$(z^2-1)(y+1)$	0	0
	$(z^2-1)(y-1)$	0	0
	0	$(z^2-1)(x+1)$	0
	0	$(z^2-1)(x-1)$	0
	0	$(x^2-1)(z+1)$	0
	0	$(x^2-1)(z-1)$	0
	0	0	$(x^2-1)(y+1)$
	0	0	$(x^2-1)(y-1)$
	0	0	$(y^2-1)(x+1)$
	0	0	$(y^2-1)(x-1)$
	$4(y^2-1)x(z+1)$	0	$(x^2-1)(y^2-1)$
	$4(y^2-1)x(z-1)$	0	$(x^2-1)(y^2-1)$
	$(y^2-1)y(z+1)$	0	0
	$(y^2-1)y(z-1)$	0	0
	$4(z^2-1)x(y+1)$	$(x^2-1)(z^2-1)$	0
	$4(z^2-1)x(y-1)$	$(x^2-1)(z^2-1)$	0
	$(z^2-1)(y+1)z$	0	0
	$(z^2-1)(y-1)z$	0	0
	0	$4(x^2-1)y(z+1)$	$(x^2-1)(y^2-1)$
	0	$4(x^2-1)y(z-1)$	$(x^2-1)(y^2-1)$
	0	$(x^2-1)x(z+1)$	0
	0	$(x^2-1)x(z-1)$	0
	$(y^2-1)(z^2-1)$	$4(z^2-1)(x+1)y$	0
	$(y^2-1)(z^2-1)$	$4(z^2-1)(x-1)y$	0
	0	$(z^2-1)(x+1)z$	0
	0	$(z^2-1)(x-1)z$	0
	0	$(x^2-1)(z^2-1)$	$4(x^2-1)(y+1)z$
	0	$(x^2-1)(z^2-1)$	$4(x^2-1)(y-1)z$
	0	0	$(x^2-1)x(y+1)$
	0	0	$(x^2-1)x(y-1)$
	$(y^2-1)(z^2-1)$	0	$4(y^2-1)(x+1)z$
	$(y^2-1)(z^2-1)$	0	$4(y^2-1)(x-1)z$
	0	0	$(y^2-1)(x+1)y$
	0	0	$(y^2-1)(x-1)y$

$S_3\Lambda^1(\square_3)$

COMPUTATIONAL FINITE ELEMENT DIFFERENTIAL FORMS (SUPPLEMENT)

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
$\mathcal{S}_1^- \Lambda^1(\square_3)$	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$
	$x(y+1)(z+1)$	0	0
	$x(y+1)(z-1)$	0	0
	$x(y-1)(z+1)$	0	0
	$x(y-1)(z-1)$	0	0
	0	$(x+1)y(z+1)$	0
$\mathcal{S}_2^- \Lambda^1(\square_3)$	0	$(x-1)y(z+1)$	0
	0	$(x+1)y(z-1)$	0
	0	$(x-1)y(z-1)$	0
	0	0	$(x+1)(y+1)z$
	0	0	$(x+1)(y-1)z$
	0	0	$(x-1)(y+1)z$
	0	0	$(x-1)(y-1)z$
	$(y^2-1)(z+1)$	0	0
	$(y^2-1)(z-1)$	0	0
	$(z^2-1)(y+1)$	0	0
	$(z^2-1)(y-1)$	0	0
	0	$(z^2-1)(x+1)$	0
	0	$(z^2-1)(x-1)$	0
	0	$(x^2-1)(z+1)$	0
	0	$(x^2-1)(z-1)$	0
	0	0	$(x^2-1)(y+1)$
	0	0	$(x^2-1)(y-1)$
	0	0	$(y^2-1)(x+1)$
	0	0	$(y^2-1)(x-1)$

	dx	dy	dz
	$(y+1)(z+1)$	0	0
	$(y+1)(z-1)$	0	0
	$(y-1)(z+1)$	0	0
	$(y-1)(z-1)$	0	0
	0	$(x+1)(z+1)$	0
	0	$(x-1)(z+1)$	0
	0	$(x+1)(z-1)$	0
	0	$(x-1)(z-1)$	0
	0	0	$(x+1)(y+1)$
	0	0	$(x+1)(y-1)$
	0	0	$(x-1)(y+1)$
	0	0	$(x-1)(y-1)$
	$x(y+1)(z+1)$	0	0
	$x(y+1)(z-1)$	0	0
	$x(y-1)(z+1)$	0	0
	$x(y-1)(z-1)$	0	0
	0	$(x+1)y(z+1)$	0
	0	$(x-1)y(z+1)$	0
	0	$(x+1)y(z-1)$	0
	0	$(x-1)y(z-1)$	0
	0	0	$(x+1)(y+1)z$
	0	0	$(x+1)(y-1)z$
	0	0	$(x-1)(y+1)z$
	0	0	$(x-1)(y-1)z$
	$x^2(y+1)(z+1)$	0	0
	$x^2(y+1)(z-1)$	0	0
	$x^2(y-1)(z+1)$	0	0
	$x^2(y-1)(z-1)$	0	0
	0	$(x+1)y^2(z+1)$	0
	0	$(x-1)y^2(z+1)$	0
	0	$(x+1)y^2(z-1)$	0
	0	$(x-1)y^2(z-1)$	0
	0	0	$(x+1)(y+1)z^2$
	0	0	$(x+1)(y-1)z^2$
	0	0	$(x-1)(y+1)z^2$
	0	0	$(x-1)(y-1)z^2$
	$(y^2-1)(z+1)$	0	0
	$(y^2-1)(z-1)$	0	0
	$(z^2-1)(y+1)$	0	0
	$(z^2-1)(y-1)$	0	0
	0	$(z^2-1)(x+1)$	0
	0	$(z^2-1)(x-1)$	0
	0	$(x^2-1)(z+1)$	0
	0	$(x^2-1)(z-1)$	0
	0	0	$(x^2-1)(y+1)$
	0	0	$(x^2-1)(y-1)$
	0	0	$(y^2-1)(x+1)$
	0	0	$(y^2-1)(x-1)$
	$(y^2-1)y(z+1)$	0	0
	$(y^2-1)y(z-1)$	0	0
	$(z^2-1)(y+1)z$	0	0
	$(z^2-1)(y-1)z$	0	0
	0	$(z^2-1)(x+1)z$	0
	0	$(z^2-1)(x-1)z$	0
	0	$(x^2-1)x(z+1)$	0
	0	$(x^2-1)x(z-1)$	0
	0	0	$(x^2-1)x(y+1)$
	0	0	$(x^2-1)x(y-1)$
	0	0	$(y^2-1)(x+1)y$
	0	0	$(y^2-1)(x-1)y$
	$(y^2-1)x(z+1)$	$-(x^2-1)y(z+1)$	0
	$(y^2-1)x(z-1)$	$-(x^2-1)y(z-1)$	0
	$(z^2-1)x(y+1)$	0	$-(x^2-1)(y+1)z$
	$(z^2-1)x(y-1)$	0	$-(x^2-1)(y-1)z$
	0	$(z^2-1)(x+1)y$	$-(y^2-1)(x+1)z$
	0	$(z^2-1)(x-1)y$	$-(y^2-1)(x-1)z$

$\mathcal{S}_3^- \Lambda^1(\square_3)$

COMPUTATIONAL FINITE ELEMENT DIFFERENTIAL FORMS (SUPPLEMENT)

	$dydz$	$dx dz$	$dx dy$
	$x + 1$	0	0
	$x - 1$	0	0
	0	$y + 1$	0
	0	$y - 1$	0
	0	0	$z + 1$
	0	0	$z - 1$
	$2(x + 1)y$	$y^2 - 1$	0
	$2(x - 1)y$	$y^2 - 1$	0
$\mathcal{S}_1 \Lambda^2(\square_3)$	$2(x + 1)z$	0	$-z^2 + 1$
	$2(x - 1)z$	0	$-z^2 + 1$
	$x^2 - 1$	$2x(y + 1)$	0
	$x^2 - 1$	$2x(y - 1)$	0
	0	$2(y + 1)z$	$z^2 - 1$
	0	$2(y - 1)z$	$z^2 - 1$
	$-x^2 + 1$	0	$2x(z + 1)$
	$-x^2 + 1$	0	$2x(z - 1)$
	0	$y^2 - 1$	$2y(z + 1)$
	0	$y^2 - 1$	$2y(z - 1)$

	$dydz$	$dx dz$	$dx dy$
	$x + 1$	0	0
	$x - 1$	0	0
	0	$y + 1$	0
	0	$y - 1$	0
	0	0	$z + 1$
	0	0	$z - 1$
	$(x + 1)z$	0	0
	$(x - 1)z$	0	0
	$(x + 1)y$	0	0
	$(x - 1)y$	0	0
	0	$(y + 1)z$	0
	0	$(y - 1)z$	0
	0	$x(y + 1)$	0
	0	$x(y - 1)$	0
	0	0	$y(z + 1)$
	0	0	$y(z - 1)$
	0	0	$x(z + 1)$
	0	0	$x(z - 1)$
	$3(x + 1)y^2$	$(y^2 - 1)y$	0
	$3(x - 1)y^2$	$(y^2 - 1)y$	0
$\mathcal{S}_2 \Lambda^2(\square_3)$	$3(x + 1)z^2$	0	$-(z^2 - 1)z$
	$3(x - 1)z^2$	0	$-(z^2 - 1)z$
	$(x^2 - 1)x$	$3x^2(y + 1)$	0
	$(x^2 - 1)x$	$3x^2(y - 1)$	0
	0	$3(y + 1)z^2$	$(z^2 - 1)z$
	0	$3(y - 1)z^2$	$(z^2 - 1)z$
	$-(x^2 - 1)x$	0	$3x^2(z + 1)$
	$-(x^2 - 1)x$	0	$3x^2(z - 1)$
	0	$(y^2 - 1)y$	$3y^2(z + 1)$
	0	$(y^2 - 1)y$	$3y^2(z - 1)$
	$4(x + 1)yz$	$(y^2 - 1)z$	$-(z^2 - 1)y$
	$4(x - 1)yz$	$(y^2 - 1)z$	$-(z^2 - 1)y$
	$(x^2 - 1)z$	$4x(y + 1)z$	$(z^2 - 1)x$
	$(x^2 - 1)z$	$4x(y - 1)z$	$(z^2 - 1)x$
	$-(x^2 - 1)y$	$(y^2 - 1)x$	$4xy(z + 1)$
	$-(x^2 - 1)y$	$(y^2 - 1)x$	$4xy(z - 1)$
	$x^2 - 1$	0	0
	0	$y^2 - 1$	0
	0	0	$z^2 - 1$

	$dydz$	$dx dz$	$dx dy$
	$x + 1$	0	0
	$x - 1$	0	0
	0	$y + 1$	0
	0	$y - 1$	0
	0	0	$z + 1$
	0	0	$z - 1$
	$(x + 1)z$	0	0
	$(x - 1)z$	0	0
	$(x + 1)y$	0	0
	$(x - 1)y$	0	0
	0	$(y + 1)z$	0
	0	$(y - 1)z$	0
	0	$x(y + 1)$	0
	0	$x(y - 1)$	0
	0	0	$y(z + 1)$
	0	0	$y(z - 1)$
	0	0	$x(z + 1)$
	0	0	$x(z - 1)$
	$(x + 1)z^2$	0	0
	$(x - 1)z^2$	0	0
	$(x + 1)yz$	0	0
	$(x - 1)yz$	0	0
	$(x + 1)y^2$	0	0
	$(x - 1)y^2$	0	0
	0	$(y + 1)z^2$	0
	0	$(y - 1)z^2$	0
	0	$x(y + 1)z$	0
	0	$x(y - 1)z$	0
	0	$x^2(y + 1)$	0
	0	$x^2(y - 1)$	0
	0	0	$y^2(z + 1)$
	0	0	$y^2(z - 1)$
	0	0	$xy(z + 1)$
	0	0	$xy(z - 1)$
	0	0	$x^2(z + 1)$
	0	0	$x^2(z - 1)$
$S_3\Lambda^2(\square_3)$	$4(x + 1)y^3$	$(y^2 - 1)y^2$	0
	$4(x - 1)y^3$	$(y^2 - 1)y^2$	0
	$4(x + 1)z^3$	0	$-(z^2 - 1)z^2$
	$4(x - 1)z^3$	0	$-(z^2 - 1)z^2$
	$(x^2 - 1)x^2$	$4x^3(y + 1)$	0
	$(x^2 - 1)x^2$	$4x^3(y - 1)$	0
	0	$4(y + 1)z^3$	$(z^2 - 1)z^2$
	0	$4(y - 1)z^3$	$(z^2 - 1)z^2$
	$-(x^2 - 1)x^2$	0	$4x^3(z + 1)$
	$-(x^2 - 1)x^2$	0	$4x^3(z - 1)$
	0	$(y^2 - 1)y^2$	$4y^3(z + 1)$
	0	$(y^2 - 1)y^2$	$4y^3(z - 1)$
	$5(x + 1)yz^2$	$(y^2 - 1)z^2$	$-(z^2 - 1)yz$
	$5(x - 1)yz^2$	$(y^2 - 1)z^2$	$-(z^2 - 1)yz$
	$5(x + 1)y^2z$	$(y^2 - 1)yz$	$-(z^2 - 1)y^2$
	$5(x - 1)y^2z$	$(y^2 - 1)yz$	$-(z^2 - 1)y^2$
	$(x^2 - 1)xz$	$5x^2(y + 1)z$	$(z^2 - 1)x^2$
	$(x^2 - 1)xz$	$5x^2(y - 1)z$	$(z^2 - 1)x^2$
	$(x^2 - 1)z^2$	$5x(y + 1)z^2$	$(z^2 - 1)xz$
	$(x^2 - 1)z^2$	$5x(y - 1)z^2$	$(z^2 - 1)xz$
	$-(x^2 - 1)y^2$	$(y^2 - 1)xy$	$5xy^2(z + 1)$
	$-(x^2 - 1)y^2$	$(y^2 - 1)xy$	$5xy^2(z - 1)$
	$-(x^2 - 1)xy$	$(y^2 - 1)x^2$	$5x^2y(z + 1)$
	$-(x^2 - 1)xy$	$(y^2 - 1)x^2$	$5x^2y(z - 1)$
	$x^2 - 1$	0	0
	0	$y^2 - 1$	0
	0	0	$z^2 - 1$
	$(x^2 - 1)z$	0	0
	$(x^2 - 1)y$	0	0
	$(x^2 - 1)x$	0	0
	0	$(y^2 - 1)x$	0
	0	$(y^2 - 1)z$	0
	0	$(y^2 - 1)y$	0
	0	0	$(z^2 - 1)y$
	0	0	$(z^2 - 1)x$
	0	0	$(z^2 - 1)z$

COMPUTATIONAL FINITE ELEMENT DIFFERENTIAL FORMS (SUPPLEMENT)

	<u>dydz</u>	<u>dxdz</u>	<u>dx dy</u>	
$S_1^- \Lambda^2(\square_3)$	$x+1$	0	0	
	$x-1$	0	0	
	0	$y+1$	0	
	0	$y-1$	0	
	0	0	$z+1$	
	0	0	$z-1$	
	<u>dydz</u>	<u>dxdz</u>	<u>dx dy</u>	
$S_2^- \Lambda^2(\square_3)$	$x+1$	0	0	
	$x-1$	0	0	
	0	$y+1$	0	
	0	$y-1$	0	
	0	0	$z+1$	
	0	0	$z-1$	
	$(x+1)z$	0	0	
	$(x-1)z$	0	0	
	$(x+1)y$	0	0	
	$(x-1)y$	0	0	
	0	$(y+1)z$	0	
	0	$(y-1)z$	0	
	0	$x(y+1)$	0	
	0	$x(y-1)$	0	
	0	0	$y(z+1)$	
	0	0	$y(z-1)$	
	0	0	$x(z+1)$	
	0	0	$x(z-1)$	
	x^2-1	0	0	
	0	y^2-1	0	
	0	0	z^2-1	
		<u>dydz</u>	<u>dxdz</u>	<u>dx dy</u>
	$S_3^- \Lambda^2(\square_3)$	$x+1$	0	0
		$x-1$	0	0
0		$y+1$	0	
0		$y-1$	0	
0		0	$z+1$	
0		0	$z-1$	
$(x+1)z$		0	0	
$(x-1)z$		0	0	
$(x+1)y$		0	0	
$(x-1)y$		0	0	
0		$(y+1)z$	0	
0		$(y-1)z$	0	
0		$x(y+1)$	0	
0		$x(y-1)$	0	
0		0	$y(z+1)$	
0		0	$y(z-1)$	
0		0	$x(z+1)$	
0		0	$x(z-1)$	
$(x+1)z^2$		0	0	
$(x-1)z^2$		0	0	
$(x+1)yz$		0	0	
$(x-1)yz$		0	0	
$(x+1)y^2$		0	0	
$(x-1)y^2$		0	0	
0		$(y+1)z^2$	0	
0		$(y-1)z^2$	0	
0		$x(y+1)z$	0	
0		$x(y-1)z$	0	
0		$x^2(y+1)$	0	
0		$x^2(y-1)$	0	
0		0	$y^2(z+1)$	
0		0	$y^2(z-1)$	
0		0	$xy(z+1)$	
0		0	$xy(z-1)$	
0		0	$x^2(z+1)$	
0		0	$x^2(z-1)$	
x^2-1		0	0	
0		y^2-1	0	
0		0	z^2-1	
$(x^2-1)x$		0	0	
0	$(y^2-1)y$	0		
0	0	$(z^2-1)z$		
$(x^2-1)y$	$-(y^2-1)x$	0		
$(x^2-1)z$	0	$(z^2-1)x$		
0	$(y^2-1)z$	$-(z^2-1)y$		