SMAI-JCM
SMAI Journal of Computational Mathematics

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Volume 2 (2016), p. 1-17.
<http://smai-jcm.cedram.org/item?id=SMAI-JCM_2016__2__1_0>

SMAI Journal of Computational Mathematics
Vol. 2, 1-17 (2016)

A Dual–Mixed Finite Element Method for the Brinkman Problem

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Abstract. A mixed variational formulation of the Brinkman problem is presented which is uniformly well–posed for degenerate (vanishing) coefficients under the hypothesis that a generalized Poincaré inequality holds. The construction of finite element schemes which inherit this property is then considered.

Math. classification. 65N30, 65N12.

Keywords. Brinkman, Stokes, Darcy, mixed methods.

1. Introduction

The focus of this work is the development of mixed finite element schemes for the stationary Brinkman problem. Given a bounded and connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary Γ, the problem is given by

$$\begin{aligned} \alpha(x)u - \operatorname{div}\Big(\nu(x)(\nabla u + \nabla u^T) - pI\Big) &= f \qquad \text{in } \Omega, \\ \operatorname{div}(u) &= g \qquad \text{in } \Omega, \\ u &= u_\Gamma \quad\ \ \text{on } \Gamma. \end{aligned} \tag{1.1}$$

Here u is velocity, p is pressure, $\alpha(x) \geq 0$ is the dynamic viscosity divided by permeability, and $\nu(x) \geq 0$ is the effective viscosity. With the usual definition of the Lebesgue space $L^2(\Omega)$, the external force f is assumed to be in $(L^2(\Omega))^d$ and $g \in L^2(\Omega)$ satisfies $\int_\Omega g\,dx = \int_\Gamma u_\Gamma \cdot n\,ds$.

In a manner similar to dual-mixed methods for the Stokes and Navier-Stokes problems [17, 18], the work presented here constructs a dual-mixed variational formulation of (1.1) in which the fluid velocity, the fluid stress, and the deviatoric part of the velocity gradient are the primary unknowns. This construction results in a twofold saddle point problem whose coercivity is guaranteed for all meaningful α, ν under a generalized Poincaré inequality. Based on this formulation, two classes of finite element methods are presented and analyzed, one of which utilizes classical finite element spaces, while the other is based on a discrete pivoting strategy. The former method is convergent, yet sub–optimal by one order, while the latter method converges with optimal order for all physically relevant α and ν.

The Brinkman problem arises in a homogenized model of fluid flow in porous media [24, 1, 2, 4, 32]. An example physical domain for the Brinkman problem is given in Figure 1.1, in which each subdomain of Ω corresponds to a different medium and effective viscosity in (1.1).

The second author was supported in part by the National Science Foundation grant DMS–1417980 and the Alfred Sloan Foundation. The third author was supported in part by National Science Foundation grants DMS–1418991 and DMREF–1434734. This work was also supported by the NSF through the Center for Nonlinear Analysis.

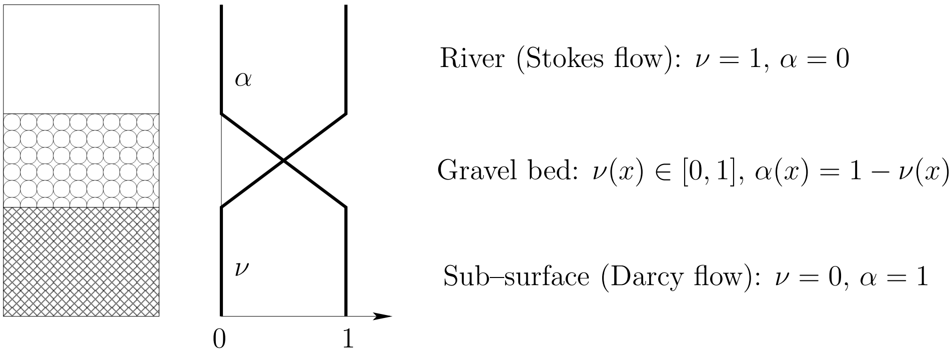

FIGURE 1.1. Typical flow configuration for the Brinkman problem.

The Brinkman model is also employed in several application areas, including filtration [28], groundwater flow through permeable membranes [10], liquid and vapor flow in heat pipes [21], flow through permeable textile microstructures [14], computational fuel cell dynamics [37], vascular tissue engineering [3], and bioreactors for tissue regeneration [30, 34].

Mathematically, the Brinkman problem (1.1) resembles both the Stokes problem for fluid flow and the Darcy problem for flow in porous media. Indeed, when $\alpha = 0$ and $\nu > 0$, the Stokes problem is recovered; alternately if $\nu = 0$ and $\alpha > 0$, (1.1) is the Darcy problem. When ν and α are both positive constants, the problem (1.1) also arises in semidiscrete formulations of the nonstationary Stokes problem [13], and in this case α represents the reciprocal of a time step. Each of these individual situations have been studied extensively, and stable finite element schemes are widely employed for computing approximate solutions. Additionally, coupled Stokes and Darcy flows, which pose different equations in subdomains of Ω with various coupling conditions, have been studied extensively (the reader is referred to [9, 20, 12, 26] for a sample of recent developments in that direction).

Constructing finite element methods to solve (1.1) for arbitrary choices of $\alpha, \nu \geq 0$ is challenging. This stems from the fundamental differences in the function spaces used to construct the variational form of (1.1) for the two extreme cases: in the Darcy limit ($\nu \to 0$), the standard mixed velocity-pressure formulation of (1.1) requires $H(\Omega; \mathrm{div}) - L^2(\Omega)$ compatibility, while in the Stokes limit ($\alpha \to 0$), the standard mixed formulation requires $H^1(\Omega) - L^2(\Omega)$ compatibility.

One approach to address this problem is to construct non-conforming finite element spaces by modifying $H(\Omega; \mathrm{div}) - L^2(\Omega)$ compatible spaces. Such efforts have been undertaken by by Mardal, Tai, and Winther [25], Tai and Winther [35] for the case $\nu \to 0$ assuming $\alpha > 0$, and Xie, Xu, and Xue [37] for the case $\nu > 0, \alpha \geq 0$. Guzmán and Neilan [15] construct nonconforming spaces for $\nu, \alpha > 0$ that satisfy mass-conservation properties. Alternately, the work of Stenberg and his collaborators [19, 16, 22, 23] approaches the Brinkman problem by modifying the mixed term in the velocity-pressure variational formulation depending on the particular value of ν (assuming $\alpha > 0$). Other approaches to the classical velocity-pressure formulation of the Brinkman problem include [38, 31]. A different approach is to consider an alternative to the classical velocity-pressure mixed variational formulation of (1.1). In this direction, Gatica, Gatica, and Márquez [11] construct a variational formulation for the Brinkman problem (with $g = 0$ and $\alpha, \nu > 0$) in which the pseudostress and the velocity are the primary unknowns.

None of the aforementioned works has utilized a single-domain variational formulation that is well–posed for all choices of $\alpha, \nu \geq 0$ satisfying $\alpha + \nu > 0$ everywhere in the problem domain. In [36], Vassilevski and Villa construct an innovative variational formulation for the Brinkman problem (with non–symmetric stress) using the vorticity, velocity, and pressure as primary unknowns and are able to show well–posedness for the constant coefficient case with $\alpha + \nu > 0$. The work presented here also addresses the case $\alpha + \nu > 0$, but does not require the coefficients to be constant and considers Dirichlet boundary conditions with both symmetric and non–symmetric constitutive laws.

The remainder of this work is organized as follows. The new dual-mixed variational formulation of equations (1.1) is established, and existence and uniqueness of solutions are shown in Section 2. In Section 3, the discrete problem is formulated and abstract criteria are given to ensure that the method is well–posed. Two finite element methods, each with different spaces and discrete pivoting strategies, are then developed. Finally, numerical examples are presented in Section 4.

2. **Variational Formulation**

Standard notation will be used for Lebesgue and Sobolev spaces, and vector or tensor spaces will be denoted using uppercase blackboard bold letters. The pairing (f, g) denotes the standard $L^2(\Omega)$ inner product for scalar and vector functions f and g. Tensor functions F and G are represented with capital letters and the Frobenius inner product is also written as (F, G). The pairing $\langle f, g\rangle$ represents a duality pairing of traces on the boundary Γ of Ω, where the dual spaces are inferred from context. The norm on the space X will be denoted by $\|\cdot\|_X$ and X^* will denote the dual space of X.

For the remainder of this work we consider the problem

$$\begin{aligned} \alpha(x)u - \operatorname{div}\left(\mathcal{A}(x, \nabla u) - pI\right) &= f \qquad \text{in } \Omega, \\ \operatorname{div}(u) &= g \qquad \text{in } \Omega, \\ u &= u_\Gamma \qquad \text{on } \Gamma. \end{aligned} \tag{2.1}$$

where the constitutive law $\mathcal{A} : \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ has the property that $\mathcal{A}(x, D) = \mathcal{A}(x, \operatorname{dev}(D))$ where $\operatorname{dev}(D)$ denotes the trace free (deviatoric) part of D,

$$\operatorname{dev}(D) = D - (1/d)\operatorname{tr}(D)I,$$

and $(G, H) \to (G, \mathcal{A}(\cdot, H))$ is a semi–inner product on $L^2(\Omega)^{d\times d}$. With this formulation our results are directly applicable to (1.1) as well as to the equation

$$\alpha(x)u - \operatorname{div}\left(\nu(x)\nabla u - pI\right) = f,$$

which is considered in several related works. Subsequently the notational dependence of α, ν, and $\mathcal{A}$ on x will be suppressed.

The dual–mixed formulation of (2.1) is derived by setting the variable $G = \operatorname{dev}(\nabla u)$ to be the trace free (deviatoric) part of the velocity gradient, and introducing the stress $S = \mathcal{A}(\nabla u) - pI$ as an additional variable. Equations (2.1) can then be written as

$$\begin{aligned} \alpha u - \operatorname{div}(S) &= f, \\ \operatorname{dev}\left(\mathcal{A}(G) - S\right) &= 0, \\ -\nabla u + G &= -(g/d)I. \end{aligned}$$

Note that if $\mathcal{A} = 0$ the middle equation requires the stress to be diagonal, $S = -p\,I$ and the pressure can be recovered via the formula $p = (1/d)\operatorname{tr}(\mathcal{A}(G) - S)$. The natural weak statement of these equations is obtained upon taking the inner product of the above with smooth functions (v, H, T), where H is trace free, and integrating the last equation by parts to get

$$\begin{aligned}(\alpha u, v) - (\operatorname{div}(S), v) &= (f, v),\\ (\mathcal{A}(G), H) - (S, H) &= 0,\\ (u, \operatorname{div}(T)) + (G, T) &= -(1/d)(g, \operatorname{tr}(T)) + \langle u_\Gamma, Tn\rangle.\end{aligned}$$

The natural spaces for the velocity and stress are

$$\mathbb{U} = L^2(\Omega)^d, \qquad \text{and} \qquad \mathbb{S} = \left\{ S \in \mathbb{H}(\operatorname{div};\Omega) \;\middle|\; \int_\Omega \operatorname{tr}(S) = 0 \right\},$$

where $\mathbb{H}(\operatorname{div};\Omega)$ denotes the functions in $L^2(\Omega)^{d\times d}$ with divergence in $L^2(\Omega)^d$. We denote the associated norms as $\|\cdot\|_{\mathbb{U}} = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{\mathbb{S}} = \|\cdot\|_{H(\operatorname{div};\Omega)}$.

To characterize the space containing the deviatoric part of the velocity gradient, first define the norm $\|\cdot\|_{\mathbb{G}}$ on $C^\infty(\bar{\Omega})^{d\times d}$ to be

$$\|G\|^2_{\mathbb{G}} := \|G\|^2_{L^2_{\mathcal{A}}(\Omega)} + \|G\|^2_{\mathbb{S}^*}, \qquad \text{where} \qquad \|G\|_{\mathbb{S}^*} := \sup_{T\in\mathbb{S}} \frac{(G,T)}{\|T\|_{\mathbb{S}}}$$

and $\|G\|^2_{L^2_{\mathcal{A}}(\Omega)} := (\mathcal{A}(G), G)$ is the semi–norm on the weighted $L^2(\Omega)$ space. Then let

$$\mathbb{G} = \overline{\{G \in C^\infty(\bar{\Omega})^{d\times d} \mid tr(G) = 0\}}^{\|\cdot\|_{\mathbb{G}}},$$

where the overbar denotes the closure with respect to the indicated norm. In this context the smooth functions are identified with elements of $\mathbb{S}^*$ by pivoting through $L^2(\Omega)^{d\times d}$ so pairings of the form (G, T) with $G \in \mathbb{G}$ and $T \in \mathbb{S}$ need to be interpreted as $G(T)$. In general $\mathbb{G} \subset \mathbb{S}^*$ and elements in $\mathbb{G}$ need not be square integrable; for example, when $\mathcal{A} = 0$ this space can be characterized as

$$\mathbb{G} = \{G \in \mathbb{H}(\Omega;\operatorname{div})^* \mid G(\phi I) = 0,\ \phi \in H^1(\Omega)\}.$$

In the other direction, if $\mathcal{A}(G) = \nu \operatorname{dev}(G)$ with $\nu \geq \nu_0 > 0$ in Ω, then

$$\mathbb{G} = \{G \in L^2(\Omega)^{d\times d} \mid tr(G) = 0\}.$$

The mixed formulation of the Brinkman problem may then be stated as $((u, G), S) \in (\mathbb{U} \times \mathbb{G}) \times \mathbb{S}$,

$$a((u, G), (v, H)) - b(S, (v, H)) = f(v, H), \qquad (v, H) \in \mathbb{U} \times \mathbb{G} \tag{2.2a}$$

$$b(T, (u, G)) = F(T), \qquad T \in \mathbb{S}, \tag{2.2b}$$

where the bilinear forms

$$a : (\mathbb{U} \times \mathbb{G}) \times (\mathbb{U} \times \mathbb{G}) \to \mathbb{R} \qquad \text{and} \qquad b : \mathbb{S} \times (\mathbb{U} \times \mathbb{G}) \to \mathbb{R}$$

are

$$a((u, G), (v, H)) = (\alpha u, v) + (\mathcal{A}(G), H) \qquad \text{and} \qquad b(T, (u, G)) = (u, \operatorname{div}(T)) + G(T)$$

respectively, $f(v, H) = (f, v)$, and $F(T) = \langle u_\Gamma, Tn\rangle - (1/d)(g, \operatorname{tr}(T))$.

Equations (2.2) pose the Brinkman equations as a standard saddle point problem, and the spaces have been chosen so that the bilinear forms are continuous and the classical inf–sup condition is satisfied. The following hypotheses on the coefficients α and $\mathcal{A}$ will guarantee coercivity of $a(\cdot,\cdot)$ over the kernel of $b(\cdot,\cdot)$.

Assumption 1 (Generalized Poincaré Inequality)**.** *There exists $c_p > 0$ such that*

$$\|u\|^2_{L^2_\alpha(\Omega)} + \|G\|^2_{L^2_{\mathcal{A}}(\Omega)} \geq c_p \|u\|^2_{L^2(\Omega)}, \tag{2.3}$$

for all

$$(u, G) \in \{(u, G) \in \mathbb{U} \times \mathbb{G} \mid (u, \operatorname{div}(T)) + G(T) = 0,\ T \in \mathbb{S}\}, \tag{2.4}$$

where $\|u\|^2_{L^2_\alpha(\Omega)} = (\alpha u, u)$.

In addition to guaranteeing that $\nabla u = G$ in the sense of distributions, condition (2.4) also implies boundary conditions upon u, the precise form depending upon $\mathcal{A}$ through the requirement $G \in \mathbb{G}$.

Example 2.1. The following prototypical examples illustrate that the generalized Poincaré inequality is satisfied for a broad class of problems which may include degeneracies. Let (u, G) be a pair satisfying (2.4) so that $\nabla u = G$ in $\mathcal{D}^*(\Omega)$, and assume that $\mathcal{A}(G) = \nu G$.

(1) If $\alpha \geq \alpha_0 > 0\ \forall x \in \Omega$, then (2.3) trivially holds with $c_p = \alpha_0$. Clearly this condition is also necessary in the degenerate case $\nu = 0$.

(2) When $\nu \geq \nu_0 > 0\ \forall x \in \Omega$ we have $\mathbb{G} \subset L^2(\Omega)^{d\times d}$ so $\nabla u = G \in L^2(\Omega)^{d\times d}$. Then $u \in H^1(\Omega)^d$ and (2.4) requires u to vanish on the boundary. The Poincaré inequality for functions in $H_0^1(\Omega)^d$ then guarantees
$\|u\|_{L^2(\Omega)} \leq (c_p/\nu_0)\|G\|_{L^2(\Omega)}$.

(3) Let $\Omega_1 = \cup_{k=0}^N D_k \subset \Omega$ be the finite union of connected, Lipschitz domains with $|\partial\Omega \cap \partial D_k| > 0$, $\nu \geq \nu_0 > 0$ on Ω_1, and $\alpha \geq \alpha_0 > 0$ on $\Omega_0 \equiv \Omega \setminus \Omega_1$ (see Figure 1.1). Appealing to the Poincaré inequality as in the previous example, it follows that

$$\|u\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\Omega_0)} + \|u\|^2_{L^2(\Omega_1)} \leq (1/\alpha_0)\|u\|^2_{L^2_\alpha(\Omega_0)} + (c_p/\nu_0)\|G\|^2_{L^2_{\mathcal{A}}(\Omega_1)}.$$

For an alternative argument that extends to the discrete setting set $u_1 = u$ on Ω_1 and zero on Ω_0. Let $T \in \mathbb{H}(\Omega_1; \mathrm{div})$ satisfy $\mathrm{div}(T) = u_1$ and $Tn = 0$ on $\bar\Omega_0 \cap \bar\Omega_1$ with $\|T\|_{\mathbb{H}(div,\Omega_1)} \leq C\|u_1\|_{L^2(\Omega)}$. Then extend T by zero to $\mathbb{H}(\Omega; \mathrm{div})$ and compute

$$\begin{aligned} \|u_1\|^2_{L^2(\Omega)} &= (u, \mathrm{div}(T)) = -(G, T) \\ &\leq \|G\|_{L^2(\Omega_1)} C \|u_1\|_{L^2(\Omega)} \\ &\leq (C/\sqrt{\nu_0})\|G\|_{L^2_{\mathcal{A}}(\Omega)}\|u_1\|_{L^2(\Omega)}. \end{aligned}$$

The next theorem shows that the dual mixed formulation developed in this section is well–posed. While we do not state it explicitly, the general theory for saddle point problems also shows that Assumption 1 is a necessary condition for the problem to be well–posed (with this choice of spaces).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and suppose that $\alpha \in L^\infty(\Omega)$ is non–negative, $\mathcal{A} : L^\infty(\Omega, \mathcal{L}(\mathbb{R}^{d\times d}))$ is the Riesz map for a semi–inner product on $L^2(\Omega, \mathbb{R}^{d\times d})$, and that Assumption 1 is satisfied. Then for each $f \in \mathbb{U}^* \times \mathbb{G}^*$ and $F \in \mathbb{S}^*$ there exists a unique $((u, G), S) \in (\mathbb{U} \times \mathbb{G}) \times \mathbb{S}$ satisfying (2.2). Moreover, there holds*

$$\|u\|_{\mathbb{U}} + \|G\|_{\mathbb{G}} + \|S\|_{\mathbb{S}} \leq C\big(\|f\|_{\mathbb{U}^*\times\mathbb{G}^*} + \|F\|_{\mathbb{S}^*}\big), \tag{2.5}$$

where $C = C(\Omega, \|\alpha\|_{L^\infty(\Omega)}, \|\mathcal{A}\|_{L^\infty(\Omega,\mathcal{L}(\mathbb{R}^{d\times d}))}, c_p) > 0$ and is otherwise independent of α and $\mathcal{A}$.

Proof. It is well known [8, 6] that problem (2.2) is well–posed and that (2.5) is satisfied if the bilinear forms are continuous, $b(\cdot,\cdot)$ satisfies an inf–sup condition and $a(\cdot,\cdot)$ is coercive over the kernel of $b(\cdot,\cdot)$. Here, the kernel of $b(\cdot,\cdot)$ is given by

$$\mathbb{Z} = \{(u, G) \in \mathbb{U} \times \mathbb{G} \mid b(T, (u, G)) = 0,\ T \in \mathbb{S}\}. \tag{2.6}$$

Continuity of the bilinear forms follows directly from the Cauchy–Schwarz inequality:

$$\begin{aligned} a((u, G), (v, H)) &\leq \|\alpha\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|G\|_{L^2_{\mathcal{A}}(\Omega)}\|H\|_{L^2_{\mathcal{A}}(\Omega)} \\ &\leq C\|(u, G)\|_{\mathbb{U}\times\mathbb{G}}\|(v, H)\|_{\mathbb{U}\times\mathbb{G}}, \end{aligned}$$

$$b(T, (u, G)) \leq \|u\|_{L^2(\Omega)}\|\mathrm{div}(T)\|_{L^2(\Omega)} + \|G\|_{\mathbb{S}^*}\|T\|_{\mathbb{S}} \leq C\|(u, G)\|_{\mathbb{U}\times\mathbb{G}}\|T\|_{\mathbb{S}}.$$

In the current setting, the inf–sup condition on $b(\cdot,\cdot)$ reads

$$\sup_{(u,G)\in\mathbb{U}\times\mathbb{G}} \frac{(u,\mathrm{div}(T))+G(T)}{\|(u,G)\|_{\mathbb{U}\times\mathbb{G}}} \geq C\|T\|_{H(\Omega;\mathrm{div})}, \qquad T\in\mathbb{S}. \tag{2.7}$$

Equivalent conditions for an inf–sup condition to hold over a product space were developed in [17] where it was shown that (2.7) holds and that $\|u\|_{\mathbb{U}} \leq C\|G\|_{\mathbb{G}}$ for pairs $(u,G)\in\mathbb{Z}$ if and only if [17, Lemma 3.2]

$$\sup_{T\in\mathbb{S}} \frac{(u,\mathrm{div}(T))}{\|T\|_{\mathbb{S}}} \geq C\|u\|_{\mathbb{U}}, \qquad \forall u\in\mathbb{U}, \qquad \text{and} \tag{2.8}$$

$$\sup_{G\in\mathbb{G}} \frac{G(T)}{\|G\|_{\mathbb{G}}} \geq C\|T\|_{\mathbb{S}}, \qquad T\in\{T\in\mathbb{S} \mid (u,\mathrm{div}(T))=0,\ u\in\mathbb{U}\}. \tag{2.9}$$

Since $\mathrm{div}:\mathbb{S}\to\mathbb{U}$ is surjective and is continuous, it follows from the closed graph theorem that $\mathrm{div}:\mathbb{S}/\mathrm{Ker}(\mathrm{div})\to\mathbb{U}$ has a continuous inverse; thus (2.8) is satisfied. To verify the second condition, recall that if $T\in\mathbb{S}$ and $\mathrm{div}(T)=0$ then $\|T\|_{\mathbb{S}} \leq C\|\mathrm{dev}(T)\|_{L^2(\Omega)}$ [5, Lemma 3.1]. Since $\mathbb{G}$ contains the trace free functions in $L^2(\Omega)^{d\times d}$, it follows that

$$\sup_{G\in\mathbb{G}} \frac{G(T)}{\|G\|_{L^2(\Omega)}} \geq \sup_{\substack{G\in L^2(\Omega)^{d\times d}\\ tr(G)=0}} \frac{(G,T)}{\|G\|_{L^2(\Omega)}} \geq \frac{(\mathrm{dev}(T),T)}{\|\mathrm{dev}(T)\|_{L^2(\Omega)}} = \|\mathrm{dev}(T)\|_{L^2(\Omega)}.$$

The second inf–sup condition (2.9) then follows since $\|G\|_{\mathbb{G}} \leq (1+\|\mathcal{A}\|_{L^\infty(\Omega,\mathcal{L}(\mathbb{R}^{d\times d}))})\|G\|_{L^2(\Omega)}$ for all $G\in\mathbb{G}\cap L^2(\Omega)^{d\times d}$. Thus the inf–sup condition (2.7) is satisfied.
To establish coercivity of $a(\cdot,\cdot)$ on the kernel note that if $(u,G)\in\mathbb{Z}$ then (1) the generalized Poincaré inequality holds for this pair, (2) $\|G\|_{\mathbb{S}^*} \leq \|u\|_{\mathbb{U}}$, and (3) $\|u\|_{\mathbb{U}} \leq C\|G\|_{\mathbb{G}}$. It then suffices to show coercivity over $\mathbb{G}$. Applying Assumption 1 yields

$$\begin{aligned} a((u,G),(u,G)) &= \|u\|^2_{L^2_\alpha(\Omega)} + \|G\|^2_{L^2_{\mathcal{A}}(\Omega)} \\ &\geq (c_p/2)\|u\|^2_{\mathbb{U}} + (1/2)\|G\|^2_{L^2_{\mathcal{A}}(\Omega)} \\ &\geq (c_p/2)\|G\|^2_{\mathbb{S}^*} + (1/2)\|G\|^2_{L^2_{\mathcal{A}}(\Omega)} \geq \min(c_p/2,1/2)\|G\|^2_{\mathbb{G}}. \end{aligned}$$

■

3. Discrete Problem

In the previous section it was essential to identify a norm for the deviatoric part of the velocity gradient for which $a(\cdot,\cdot)$ would be coercive on the kernel of $b(\cdot,\cdot)$. This was facilitated by identifying (trace free) smooth functions as a dense subspace of $\mathbb{S}^*$ by pivoting through $L^2(\Omega)^{d\times d}$. With this strategy both of the pairings $H^*(S)=(H,S)$ and $(\mathcal{A}(G),H)$ are well defined for $S\in\mathbb{S}$ and $G\in L^2_{\mathcal{A}}(\Omega)$.

If $\mathbb{G}_h\subset\mathbb{G}$ and $\mathbb{S}_h\subset\mathbb{S}$ are closed subspaces we may select a pivoting strategy $*:\mathbb{G}_h\to L^2(\Omega)^{d\times d}$ so that $\mathbb{G}_h$ has optimal approximation properties in $L^2(\Omega)$ and $\mathbb{G}^*_h \equiv \{G^*_h \mid G_h\in\mathbb{G}_h\}$ has optimal approximation properties for the dual space. In this context, the numerical scheme becomes $(u_h,G_h,S_h)\in\mathbb{U}_h\times\mathbb{G}_h\times\mathbb{S}_h$

$$a((u_h,G_h),(v_h,H^*_h)) - b((v_h,H^*_h),S_h) = f(v_h,H^*_h) \tag{3.1a}$$

$$b((u_h,G^*_h),T_h) = 0, \tag{3.1b}$$

for all $(v_h, H_h^*, T_h) \in \mathbb{U}_h \times \mathbb{G}_h^* \times \mathbb{S}_h$. If $(\mathcal{A}(\cdot, G_h), G_h^*) \geq 0$ the natural norm on $\mathbb{G}_h$, for which $a(\cdot,\cdot)$ will be coercive on the discrete kernel, is

$$\|G_h\|_{\mathbb{G}_h}^2 = (\mathcal{A}(G_h), G_h^*) + \|G_h\|_{\mathbb{S}_h^*}^2, \qquad \text{where} \qquad \|G_h\|_{\mathbb{S}_h^*} = \sup_{T_h \in \mathbb{S}_h} \frac{G_h^*(T_h)}{\|T_h\|_{\mathbb{S}}}. \tag{3.2}$$

In general this may only be a semi–norm; however, in the next theorem we require it to be a norm on the space $\mathbb{G}_h$. The proof of the following theorem is identical to that of Theorem 2.2 so is omitted.

Theorem 3.1. *Let $\mathbb{U}_h \subset \mathbb{U}$, $\mathbb{S}_h \subset \mathbb{S}$ and $\mathbb{G}_h \subset \mathbb{G}$ be closed subspaces, and let $* : \mathbb{G}_h \to L^2(\Omega)^{d\times d}$ be an embedding. Let $\|\cdot\|_{\mathbb{G}_h}$ be given by (3.2) and $\mathbb{Z}_h$ denote the discrete kernel of $b(\cdot,\cdot)$:*

$$\mathbb{Z}_h = \{(u_h, G_h) \in \mathbb{U}_h \times \mathbb{G}_h \mid b(T_h, (u_h, G_h^*)) = 0,\ T_h \in \mathbb{S}_h\}.$$

Assume that the coefficients and data satisfy the hypotheses of Theorem 2.2 and that the discrete spaces satisfy the following properties.

(1) *Discrete Poincaré inequality: The pairing $(\mathcal{A}(G_h), G_h^*) \geq 0$ and there exists $c_p > 0$ such that*

$$\|u_h\|_{L^2_\alpha(\Omega)}^2 + (\mathcal{A}(G_h), G_h^*) \geq c_p \|u_h\|_{\mathbb{U}}^2, \qquad (u_h, G_h) \in \mathbb{Z}_h.$$

(2) *Discrete dual norm: $\|\cdot\|_{\mathbb{G}_h}$ is a norm on $\mathbb{G}_h$. In the degenerate setting ($\mathcal{A} = 0$) this requires $\sup_{T_h \in \mathbb{S}_h} G_h^*(T_h) > 0$ for all $G_h \in \mathbb{G}_h \setminus \{0\}$.*

(3) *Discrete inf–sup condition: There exist constants $c_1, c_2 > 0$ such that*

$$\sup_{T_h \in \mathbb{S}_h} \frac{(u_h, \operatorname{div}(T_h))}{\|T_h\|_{\mathbb{S}}} \geq c_1 \|u_h\|_{\mathbb{U}}, \qquad u_h \in \mathbb{U}_h, \quad \textit{and} \tag{3.3}$$

$$\sup_{G_h \in \mathbb{G}_h} \frac{(G_h^*, T_h)}{\|G_h\|_{\mathbb{G}_h}} \geq c_2 \|T_h\|_{\mathbb{S}}, \qquad T_h \in \{T_h \in \mathbb{S}_h : (u_h, \operatorname{div}(T_h)) = 0,\ u_h \in \mathbb{U}_h\}. \tag{3.4}$$

Then for each $f \in \mathbb{U}^* \times \mathbb{G}_h^*$, and $F \in \mathbb{S}^*$ the discrete weak problem (3.1) has a unique solution. Moreover, there is a constant $C > 0$ depending on c_p, c_1, c_2 such that

$$\|u_h\|_{\mathbb{U}} + \|G_h\|_{\mathbb{G}_h} + \|S_h\|_{\mathbb{S}} \leq C(\|f\|_{\mathbb{U}_h^* \times \mathbb{G}_h^*} + \|F\|_{\mathbb{S}_h^*}). \tag{3.5}$$

The error estimate for this scheme does not follow immediately from the general saddle point theory since $\|\cdot\|_{\mathbb{S}^*} \neq \|\cdot\|_{\mathbb{S}_h^*}$ and the pivoting strategies for the continuous and discrete schemes may differ. When $\mathbb{G}_h \subset L^2(\Omega)^{d\times d}$ the orthogonality condition satisfied by solutions of the discrete problem gives the estimate

$$\begin{aligned} &\|u_p - u_h\|_{\mathbb{U}} + \|G_p - G_h\|_{\mathbb{G}_h} + \|S_p - S_h\|_{\mathbb{S}} \\ &\qquad \leq C\Big(\|u_p - u\|_{\mathbb{U}} + \|G_p - G\|_{L^2_{\mathcal{A}}(\Omega)} + \|\operatorname{div}(S_p - S)\|_{L^2(\Omega)} \\ &\qquad\qquad + \|G_p^* - G\|_{\mathbb{S}_h^*} + \sup_{H_h \in \mathbb{G}_h} \frac{|(S_p - S, H_h^*)|}{\|H_h\|_{\mathbb{G}_h}}\Big) \end{aligned} \tag{3.6}$$

for all $(u_p, G_p, S_p) \in \mathbb{U}_h \times \mathbb{G}_h \times \mathbb{S}_h$.

The following two sections address the construction of spaces satisfying the hypotheses of this theorem and the corresponding error estimate that results from this formula. While the discrete Poincaré is method specific, the argument presented in the third case of Example 2.1 extends to the discrete settings, so the Poincaré inequality will simply be assumed below.

3.1. **Classical Elements & Pivoting**

There are many pairs $(\mathbb{U}_h, \mathbb{S}_h) \subset L^2(\Omega)^d \times \mathbb{H}(\Omega; \mathrm{div})$ satisfying the first inf–sup condition; the Raviart–Thomas space

$$(\mathbb{U}_h, \mathbb{S}_h) = \mathcal{P}_k(\mathcal{T}_h)^d \times \mathbb{RT}_k(\mathcal{T}_h)^d \tag{3.7a}$$

on simplices and product elements being prototypical [33, 29]. Using the natural pivoting strategy, $G_h^*(T_h) = (G_h, T_h)_{L^2(\Omega)^{d\times d}}$, and selecting

$$\mathbb{G}_h = \mathrm{dev}(\mathcal{P}_k(\mathcal{T}_h)^{d\times d}) \equiv \{G_h \in \mathcal{P}_k(\mathcal{T}_h)^{d\times d} \mid \mathrm{tr}(G_h) = 0\}, \tag{3.7b}$$

the second inf–sup condition (3.4) follows a fortiori from the continuous case since $\mathrm{div} : \mathbb{S}_h \to \mathbb{U}_h$ is surjective.

3.1.1. *Non–Degenerate Problem*

When $\mathcal{A} \geq \nu_0 I > 0$ it is immediate that $\|\cdot\|_{\mathbb{G}_h} \simeq \|\cdot\|_{L^2(\Omega)}$ and Theorem 3.1 establishes coercivity of the weak form and existence and well–posedness of discrete solutions follows. In this situation the orthogonality condition (3.6) gives the error estimate

$$\begin{aligned}
&\|u - u_h\|_{\mathbb{U}} + \|G - G_h\|_{L^2(\Omega)} + \|S - S_h\|_{\mathbb{S}} \\
&\leq C \inf_{(v_h, H_h, T_h) \in \mathbb{U}_h \times \mathbb{G}_h \times \mathbb{S}_h} \left(\|u - v_h\|_{\mathbb{U}} + \|G - H_h\|_{L^2(\Omega)} + \|S - T_h\|_{\mathbb{S}} \right) \\
&\leq Ch^{k+1} \left(\|u\|_{H^{k+1}(\Omega)} + \|G\|_{H^{k+1}(\Omega)} + \|S\|_{H^{k+1}(\mathrm{div}, \Omega)} \right).
\end{aligned}$$

3.1.2. *Degenerate Case*

When $\mathcal{A}$ vanishes on a portion of the domain we do not have a proof that $\|\cdot\|_{\mathbb{G}_h}$ is a norm in spite of numerical experiments that suggest that it is. If $\mathcal{T}_h$ is a simplicial partition, a calculation shows $\dim(\mathbb{G}_h) \leq \dim(\mathbb{S}_h)$, so it is plausible that the choice of spaces (3.7) satisfies the first hypothesis of Theorem 3.1 independently of the strict positivity of $\mathcal{A}$.

In the absence of a convenient characterization of $\|\cdot\|_{\mathbb{G}_h}$, bounding the last term in the orthogonality condition (3.6) is problematic. In the degenerate case $\mathcal{A} = 0$ the stress is diagonal with $S = Ip$ and the last term in (3.6) vanishes upon selecting $S_p = p_h I$ where $p_h \in \mathcal{P}_k(\mathcal{T}_h) \cap H^1(\Omega)$ is the classical Lagrange interpolant of p. This results in the sub–optimal estimate

$$\begin{aligned}
&\|u - u_h\|_{\mathbb{U}} + \|G - G_h\|_{\mathbb{G}_h} + \|S - S_h\|_{\mathbb{S}} \\
&\qquad \leq Ch^k \left(\|u\|_{H^k(\Omega)} + \|G\|_{H^k(\Omega)} + \|p\|_{H^{k+1}(\Omega)} \right).
\end{aligned}$$

Numerical experiments confirm this rate; in particular, the method with spaces (3.7) is not convergent in the lowest order case, $k = 0$, unless $\mathcal{A} \geq \nu_0 I > 0$.

3.2. **Alternative Pivoting**

The denominator in the last term of the orthogonality condition (3.6) is only implicitly defined through equation (3.2), and it may be unclear whether it actually is a norm, and if so how to estimate it. To circumvent this issue, we construct a specific choice of spaces $\mathbb{G}_h$ and pivoting strategy $* : \mathbb{G}_h \to \mathbb{S}_h^*$ for which $\|\cdot\|_{\mathbb{S}_h^*}$ is a norm on $\mathbb{G}_h$, and a projection $S_p \in \mathbb{S}_h$ with optimal approximation properties for which the numerator of this last term in (3.6) vanishes. The developments in this section are motivated by the following lemmas, the proofs of which are given in the appendix.

Lemma 3.2. *If $u \in H^1(\Omega)$ and $u_p \in \mathbb{RT}_k(\mathcal{T}_h)$ is the Raviart–Thomas projection, then by writing $G = \operatorname{dev}(\nabla u)$,*

$$\Big| \sum_{K\in\mathcal{T}_h} (G - \operatorname{dev}(\nabla u_p), T_h)_K + \sum_{e\in\mathcal{E}_h} \int_e T_h : ([u_p] \otimes n_e)\, ds \Big| \qquad (3.8)$$
$$\leq C\|u - u_p\|_{H(\Omega;\operatorname{div})} \|T_h\|_{H(\Omega;\operatorname{div})},$$

for all $T_h \in \mathbb{RT}_k(\mathcal{T}_h)^d$. Here, the jump on an interior edge is $[u_p] = u_p^+ - u_p^-$ with $u_p^\pm(x) = \lim_{s\searrow 0} u_p(x \pm s n_e)$ where n_e is a normal to the edge and the jump on a boundary edge is $[u_p] = u_p - P_{k,e}(u)$ where $P_{k,e} : L^2(e)^d \to \mathcal{P}_k(e)$ is orthogonal projection.

Since the jump in the normal component of the Raviart–Thomas projection vanishes, this lemma shows that (for sufficiently regular G) (G, T_h) can be well-approximated by functions $G_p^* \in \mathbb{S}_h^*$ of the form

$$G_p^*(T_h) = \sum_{K\in\mathcal{T}_h} (G_K, T_h)_K + \sum_{e\in\mathcal{E}_h} \int_e T_h : (g_e \otimes n_e),$$

where $G_K \in \mathcal{P}_k(K)^{d\times d}$ and $g_e \in \mathcal{P}_k(e)^d_{\tan}$ is a vector valued polynomial tangent to e. The following lemma shows that these degrees of freedom are sufficient to control the deviatoric part of T_h required for the second inf–sup condition (3.4).

Lemma 3.3. (1) *If K is a simplex, then every $H_h \in \operatorname{dev}(\mathcal{P}_k(K)^{d\times d})$ is uniquely determined by*

- (a) *The moments of $(I - n_e \otimes n_e)H_h n_e$ up to degree k on each $e \subset \partial K$.*
- (b) *The moments of H_h up to degree $(k-1)$ on K.*

(2) *Let $\mathbb{RT}_k(K)$ be the local Raviart–Thomas space. Then if K is a cube, the deviatoric part of every divergence–free $T_h \in \mathbb{RT}_k(K)^d$ is uniquely determined by*

- (a) *The moments of $(I - n_e \otimes n_e)T_h n_e$ up to degree k on each $e \subset \partial K$.*
- (b) *The values $(T_h, \operatorname{dev}(\kappa))_K$ for all divergence–free and diagonal $\kappa \in \mathbb{P}_k(K)^{d\times d}$.*

Given G_p^* as above, it is necessary to construct a function G_p which approximates $G = \operatorname{dev}(\nabla u)$ in order to compute an approximation of $\mathcal{A}(G)$. That is, to identify an inverse of the maps $u \to u_p \to G_p^*$. To illustrate how this idea can be used to construct finite element spaces, we explicitly construct the lowest order elements on rectangular meshes. While it is possible to construct more general spaces for polynomials of higher degree on more general meshes, these developments are long and technical.

3.2.1. *Low-Order Elements on Rectangular Meshes*

Letting $\mathcal{T}_h$ be a rectangular mesh we consider consider pairs $(\mathbb{U}_h, \mathbb{S}_h) = \mathcal{P}_0(\mathcal{T}_h)^2 \times \mathbb{RT}_0(\mathcal{T}_h)^2$ (which coincides with the $BDFM_1$ space [7]). If $u \in \mathcal{P}_1(\mathbb{R}^2)^2$ and $u_p \in \mathbb{RT}_0(\mathcal{T}_h)$ is its Raviart–Thomas projection then $(u - u_p)(x) = (u - u_p)(x_K) + \operatorname{dev}(\nabla u)(x - x_K)$ on each rectangle $K \in \mathcal{T}_h$, and a calculation shows that

$$(\nabla u) : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = (\nabla u_p) : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad (\nabla u) : (n_e^\perp \otimes n_e) = \frac{2}{h_{e+}^\perp + h_{e-}^\perp} [u_p] \cdot n_e^\perp$$

on each internal edge e. Here n_e is a normal to the edge and $n_e^\perp$ the tangential vector obtained by rotating n_e by $\pi/2$, and $h_{e\pm}^\perp$ denotes the perpendicular height of the adjacent rectangles. Given degrees of freedom $\{g_e\}_{e\in\mathcal{E}_h} \cup \{g_K\}_{K\in\mathcal{T}_h}$ the lowest order element with these degrees of freedom ($G_p(x_e) : (n_e^\perp \otimes n_e) = g_e$ etc.) is

$$G_p|_K = \begin{bmatrix} g_K/2 & -\frac{g_B(y_T - y) + g_T(y - y_B)}{y_T - y_B} \\ \frac{g_L(x_R - x) + g_R(x - x_L)}{x_R - x_L} & -g_K/2 \end{bmatrix} \equiv g_K \Psi_K + \sum_{e\subset\partial K} g_e \Psi_e,$$

where the edges and coordinates of K are labeled (T)op, (B)ottom, (L)eft, (R)ight. For $T_h \in \mathbb{S}_h$ let

$$
\begin{aligned}
(G_p^*, T_h) &= \sum_{K\in\mathcal{T}_h} g_K|K| \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : T_h(x_K) + \sum_{e\in\mathcal{E}_h}(g_e/2)h_e(h_{e+}^\perp + h_{e-}^\perp)(n_e^\perp\otimes n_e) : T_h(x_e)\\
&= \sum_{K\in\mathcal{T}_h}|K|\left(g_K\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} : T_h(x_K) + \sum_{e\subset\partial K}(g_e/2)(n_e^\perp\otimes n_e) : T_h(x_e)\right)\\
&\equiv \sum_{K\in\mathcal{T}_h}\left(g_K\Psi_K^* + \sum_{e\subset\partial K} g_e\Psi_e^*, T_h\right)_K,
\end{aligned}
$$

where $x_K \in K$ and $x_e \in e$ are the centroids. If $u \in H^1(\mathrm{div};\Omega)$ and g_e is the average of $(\nabla u) : (n_e^\perp \otimes n_e)$ on e and g_K the average of $(\nabla u) : diag(1,-1)$ over K, then Lemma 3.2 shows that $\|\mathrm{dev}(\nabla u) - G_p^*\|_{\mathbb{S}_h^*} = O(h)$ and $\|\mathrm{dev}(\nabla u) - G_p\|_{L^2(\Omega)} = O(h)$, since $\mathrm{dev}(\nabla u) - G_p$ vanishes when u is linear on a neighborhood of K.

The dual function G_p^* on each element is

$$
G_p^*|_K = \begin{bmatrix} g_K & -g_B\left(\frac{6(y_T-y)}{y_T-y_B}-2\right) - g_T\left(\frac{6(y-y_B)}{y_T-y_B}-2\right)\\ g_L\left(\frac{6(x_R-x)}{x_R-x_L}-2\right) + g_R\left(\frac{6(x-x_L)}{x_R-x_L}-2\right) & -g_K\end{bmatrix},
$$

and the basis functions Ψ_K^* and $\{\Psi_e^*\}_{e\subset\partial K} \subset L^2(K)^{2\times 2}$ are immediate. Using these formulas, the hypotheses of Theorem 3.1 are readily verified.

(1) We construct non–negative approximations of $(\mathcal{A}(G_h), G_h^*)$ for the two prototypical constitutive laws in equations (1.1) and (2.1).

   (a) $\mathcal{A}(x,G) = \nu(x)G$: The dual basis functions were constructed so that
   $$(G_h, G_h^*)_K = |K|(g_K^2 + \sum_{e\subset\partial K} g_e^2).$$
   In particular,
   $$2\|G_h\|^2_{L^2(K)} \le (G_h, G_h^*)_K \le 6\|G_h\|^2_{L^2(K)}.$$
   Thus if $(\mathcal{A}(\cdot,G_h), G_h^*)$ is approximated by $(\bar\nu G_h, G_h^*)_K$, where $\bar\nu_K = \bar\nu|_K = \sup_{x\in K}\nu(x)$, it is immediate that this quantity is non–negative when $\nu(\cdot) \ge 0$.

   (b) $\mathcal{A}(x,G) = \nu(x)(G+G^\top)$: Writing G_h^{sym} for the symmetric part of G_h, a calculation shows
   $$2\|G_h^{sym}\|^2_{L^2(K)} \le (G_h^{sym}, G_h^*)_K \le 6\|G_h^{sym}\|^2_{L^2(K)}.$$
   Approximating $(\mathcal{A}(\cdot,G_h), G_h^*)$ by $2\bar\nu(G_h^{sym}, G_h^*)$, it is immediate that this quantity is non–negative when $\nu(\cdot) \ge 0$.

(2) We show that $\|\cdot\|_{\mathbb{S}_h^*}$ is a norm on $\mathbb{G}_h$. To this end, suppose that $G_h \in \mathbb{G}_h$ satisfies $(G_h^*, T_h) = 0$ for all $T_h \in \mathbb{S}_h$. Since $T_h|_K \in \mathcal{P}_1(K)^{d\times d}$ for each rectangle K it follows that
$$
\begin{aligned}
T_h(x_K) : \mathrm{diag}(1,-1) &= T_h(x_K) : (e_1\otimes e_1 - e_2\otimes e_2)\\
&= (1/2)(T_h(x_L)+T_h(x_R)) : e_1\otimes e_1 - (1/2)(T_h(x_T)+T_h(x_B)) : e_2\otimes e_2,
\end{aligned}
$$
where $e_1 = (1,0)^\top$ and $e_2 = (0,1)^\top$ are the standard basis vectors of $\mathbb{R}^2$. It follows that $T_h(x_K) : \mathrm{diag}(1,-1)$ vanishes if the normal degrees of freedom $T_h(x_e) : n_e\otimes n_e$ vanish. Thus selecting the degrees of freedom for T_h to be $T_h(x_e) : n_e\otimes n_e = 0$ and $T_h(x_e) : n_e\otimes n_e^\perp = g_e$ shows
$$0 = (G_h^*, T_h) = \sum_K |K| \sum_{e\subset\partial K} |g_e|^2,$$

and therefore the edge degrees of freedom vanish.

To verify that the face degrees of freedom vanish, let $\hat{\Omega} = \cup\{K \mid g_K \neq 0\}$. If $\hat{\Omega}$ is non–empty, let $\hat{K} \subset \hat{\Omega}$ be a boundary rectangle of $\hat{\Omega}$, and let $\hat{e} \subset \partial\hat{K} \cap \partial\hat{\Omega}$ be an edge on the boundary. Selecting T_h to have $T_h(x_{\hat{e}}) : n_{\hat{e}} \otimes n_{\hat{e}} = 1$ and all other degrees of freedom to vanish gives the contradiction.

$$0 = (G_h^*, T_h) = (\pm 1/2) g_{\hat{K}} \quad \Rightarrow \hat{K} \notin \hat{\Omega}.$$

(The sign depending upon whether $\hat{e}$ is a horizontal or vertical edge.)

(3) The BDFM family of elements were constructed so that the first inf–sup condition (3.3) is satisfied. Lemma 3.3 is used to verify second inf–sup condition (3.4). Selecting the degrees of freedom for G_h^* to be as in the lemma immediately gives $\|\mathrm{dev}(T_h)\|^2_{L^2(\Omega)} \leq (G_h^*, T_h)$. The second inf–sup then follows since $\|G_h\|_{\mathbb{G}_h} \leq C\|G_h\|_{L^2(\Omega)} \leq C\|\mathrm{dev}(T_h)\|_{L^2(\Omega)}$.

Equation (3.6) can now be used to establish a first order rate of convergence with constants depending upon α and ν only through the generalized Poincaré constant as follows.

- The estimate $\|u - u_p\|_{L^2(\Omega)} \leq Ch\|u\|_{H^1(\mathrm{div},\Omega)}$ is immediate when u_p is the Raviart–Thomas projection of u.

- The spaces $\mathbb{G}_h$ and $\mathbb{G}_h^*$ and projections in this section were constructed so that
$$\|G_p^* - G\|_{\mathbb{S}_h^*} \leq C\|u - u_p\|_{H(\Omega;\mathrm{div})} \leq Ch\|u\|_{H^1(\mathrm{div},\Omega)}.$$
The piecewise constant approximation $\bar{\nu}$ of the viscosity ν used to construct non–negative approximations of $(\mathcal{A}(G_h), G_h^*)$ gives rise to an additional consistency error of the form
$$\|(\nu/\sqrt{\bar{\nu}} - \sqrt{\bar{\nu}})G\|_{L^2(\Omega)} \leq \|\nu/\sqrt{\bar{\nu}} - \sqrt{\bar{\nu}}\|_{L^2(\Omega)}\|G\|_{L^\infty(\Omega)}.$$
For the prototypical situation illustrated in Figure 1.1 an explicit calculation shows
$$\|\nu/\sqrt{\bar{\nu}} - \sqrt{\bar{\nu}}\|_{L^2(\Omega)} \leq Ch\sqrt{\|\nabla\nu\|_{L^\infty(\Omega)}\ln(1/h)}$$
when $\bar{\nu}|_K = \nu(x_K)$ and $c\max(\nu|_K) \leq \bar{\nu}|_K \leq \max(\nu|_K)$ on an element where ν vanishes.

- We select S_p so that the last term in (3.6) vanishes. The dual basis functions were selected so that $(T_h, \Psi_e^*) = T_h(x_e) : (n_e^\perp \otimes n_e)$ for any piecewise linear function on $\mathcal{T}_h$. Let $\{\Phi_e^*\} \subset \mathcal{P}_1(\mathcal{T}_h)^{2\times 2}$ be the analogous dual basis for which $(T_h, \Phi_e^*) = T_h(x_e) : n_e \otimes n_e$ and select the tangential and normal degrees of freedom of S_p to be (S, Ψ_e^*) and (S, Φ_e^*) respectively. As in the proof of the second inf–sup condition above, it follows that $(S_p, \Psi_K^*) = (S, \Psi_K^*)$ so the last term in term (3.6) vanishes. It is immediate that $S - S_p$ vanishes when when $S \in \mathcal{P}_1(\mathbb{R}^d)^{2\times 2}$, and so $\|\mathrm{div}(S - S_p)\|_{L^2(\Omega)} \leq Ch\|S\|_{H^2(\Omega)}$.

4. **Numerical Examples**

We manufacture solutions of equations (1.1) on $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ with data f, g, and u_Γ chosen so that
$$u = \mathrm{curl}(\psi) + \nabla\phi,$$
where $\mathrm{curl}(\psi) = (-\psi_y, \psi_x)^T$,
$$\begin{aligned} \psi &= e^{x^2+y}\sin^2(2\pi x)\sin^2(2\pi y), \\ \phi &= e^{y^2+x}\cos^2(2\pi x)\cos^2(2\pi y), \end{aligned}$$

Table 4.1. Non–symmetric stress $\mathcal{A}(G) = \nu G$.

h	$\|u - u_h\|_{L^2}$	$\|G - G_h\|_{L^2_{\mathcal{A}}}$	$\|G - G_h\|_{\mathbb{S}_h^*}$	$\|S - S_h\|_{L^2}$	$\|S - S_h\|_{\mathbb{H}(\mathrm{div})}$
1	46.19	267.77	89.37	586.31	4005.05
1/2	15.34	262.45	37.83	329.48	3260.96
1/4	7.85	123.02	3.17	97.24	1642.23
1/8	2.38	35.66	0.42	28.53	620.59
1/16	0.61	9.43	0.05	7.21	163.36
1/32	0.15	2.39	0.01	1.81	41.37
Rate	1.99	1.98	2.98	1.99	1.98
Norm	23.11	284.31		327.55	5068.74

Table 4.2. Symmetric stress $\mathcal{A}(G) = \nu(G + G^\top)$.

h	$\|u - u_h\|_{L^2}$	$\|G - G_h\|_{L^2_{\mathcal{A}}}$	$\|G - G_h\|_{\mathbb{S}_h^*}$	$\|S - S_h\|_{L^2}$	$\|S - S_h\|_{\mathbb{H}(\mathrm{div})}$
1	67.69	391.62	101.02	1130.40	5713.53
1/2	42.37	355.70	51.70	701.59	5412.97
1/4	9.11	161.94	5.55	235.28	2602.92
1/8	4.60	73.65	3.96	132.78	994.20
1/16	2.26	33.50	2.17	64.94	261.69
1/32	1.12	16.25	1.11	32.24	66.28
Rate	1.00	1.04	0.96	1.01	1.98
Norm	23.11	336.26		576.17	8153.62

Errors and rates for non–degenerate examples on triangles, $\nu = \alpha = 1$.

and

$$p = e^{xy}\cos(2\pi x)\sin(2\pi y) + C.$$

The constant C is chosen so that $\int_\Omega tr(S) = 0$ and depends upon the choice of $\mathcal{A}$. We first consider a non–degenerate case with $\alpha = \nu = 1$, and then a degenerate case for which (see Figure 1.1)

$$\nu(x, y) = \begin{cases} 1 & y \geq 1/2 \\ y + 1/2 & -1/2 \leq y \leq 1/2 \\ 0 & y \leq -1/2 \end{cases} \qquad \text{and} \qquad \alpha = 1 - \nu, \tag{4.1}$$

with the symmetric and the non–symmetric constitutive laws. Errors for G are presented for each of the (semi) norms $\|G\|_{L^2_{\mathcal{A}}(\Omega)} = (\mathcal{A}(G), G)^{1/2}$ and $\|G\|_{\mathbb{S}_h^*}$. The latter is mesh dependent and computed as $\|G\|_{\mathbb{S}_h^*} = \|\mathcal{R}_h(G)\|_{H(\Omega;\mathrm{div})}$ where $\mathcal{R}_h : \mathbb{S}_h^* \to \mathbb{S}_h$ is the Riesz map;

$$\mathcal{R}_h(G) \in \mathbb{S}_h, \qquad (\mathcal{R}_h(G), T_h)_{\mathbb{H}(\Omega;\mathrm{div})} = G(T_h), \qquad T_h \in \mathbb{S}_h.$$

In the tables to follow, *Rate* is the observed rate of convergence for the two finest meshes in that norm or semi-norm, and *Norm* is the norm or semi-norm of the corresponding component of the exact solution.

4.1. Non-degenerate Case on Triangles

As described in Section 3.1, standard Raviart–Thomas elements for $\mathbb{S}_h$ and discontinuous Lagrange elements for $\mathbb{G}_h$ and $\mathbb{U}_h$ satisfy the inf–sup conditions (3.3)–(3.4). Moreover, if $\|\cdot\|_{\mathbb{S}_h^*}$ is a norm, then the error estimate (3.6) is satisfied.

Table 4.3. Non–symmetric stress $\mathcal{A}(G) = \nu G$.

h	$\|u - u_h\|_{L^2}$	$\|G - G_h\|_{L^2_{\mathcal{A}}}$	$\|G - G_h\|_{\mathbb{S}_h^*}$	$\|S - S_h\|_{L^2}$	$\|S - S_h\|_{\mathbb{H}(\mathrm{div})}$
1	59.10	221.05	95.69	449.08	3317.63
1/2	17.07	228.23	38.67	265.45	2575.43
1/4	8.43	102.80	4.39	75.48	1340.08
1/8	3.06	29.95	1.97	22.47	498.22
1/16	1.20	7.94	1.03	5.86	131.11
1/32	0.55	2.02	0.53	2.03	33.21
Rate	1.13	1.98	0.97	1.53	1.98
Norm	23.11	239.48		256.71	4001.66

Table 4.4. Symmetric stress $\mathcal{A}(G) = \nu(G + G^\top)$.

h	$\|u - u_h\|_{L^2}$	$\|G - G_h\|_{L^2_{\mathcal{A}}}$	$\|G - G_h\|_{\mathbb{S}_h^*}$	$\|S - S_h\|_{L^2}$	$\|S - S_h\|_{\mathbb{H}(\mathrm{div})}$
1	98.55	350.38	122.28	706.91	4476.65
1/2	59.13	307.06	65.40	550.13	3989.55
1/4	9.54	123.41	6.22	168.26	1962.87
1/8	5.04	58.46	4.46	99.34	738.22
1/16	2.46	26.57	2.38	48.52	194.06
1/32	1.22	12.90	1.21	24.22	49.14
Rate	1.01	1.04	0.97	1.00	1.98
Norm	23.11	270.91		431.71	5916.56

Errors and rates for degenerate case on triangles, ν as in (4.1) *and $\alpha = 1 - \nu$.*

The results for the case $\mathcal{A}(G) = \nu G$ are presented in Table 4.1 and exhibit optimal rates of convergence, namely, $\mathcal{O}(h^{k+1})$. These rates agree with the arguments given in Section 3.1.1 (since $\mathcal{A} \geq I$). In addition, the numerical results suggest that the error of G_h in the dual norm has order $\mathcal{O}(h^{k+2})$.

Table 4.2 presents the corresponding results in the symmetric case $\mathcal{A}(G) = (1/2)(G + G^\top)$. In this case, the rate of convergence decrease for all variables by an order of one; we observe linear convergence in the energy norms when $k = 1$. This behavior is most likely attributed to a lack of a discrete Korn inequality over the kernel $\mathbb{Z}_h$, implying that $\|\cdot\|_{\mathbb{G}_h} \not\simeq \|\cdot\|_{L^2(\Omega)}$. On the other hand, if $\|\cdot\|_{\mathbb{S}_h^*}$ is a norm on $\mathbb{G}_h$ and if the mesh is quasi–uniform, then a simple scaling argument shows that $\|H\|_{\mathbb{S}_h^*} \geq Ch\|P_{\mathbb{S}}(H)\|_{L^2(\Omega)}$, where $P_{\mathbb{S}}$ is the L^2–projection onto $\mathbb{S}$. In light of the last term in (3.6), this formal argument suggests the observed sub–optimal rates.

4.2. **Degenerate Case on Triangles**

Table 4.3 contains results for ν defined in (4.1) and $\alpha = 1 - \nu$ for $\mathcal{A}(G) = \nu G$, while Table 4.4 contains results for $\mathcal{A}(G) = \nu(G + G^\top)$. While the problem is well–posed for this degeneracy in ν, in both cases the rates of convergence degrade due to the presence of the last term on the right of (3.6).

4.3. **Degenerate Case on Rectangles**

The rectangular element developed in Section 3.2.1 exhibits the optimal first order rate of convergence for every combination of degenerate and non–degenerate coefficients and symmetric and non–symmetric constitutive law. Errors and rates for the degenerate coefficients are presented in Tables 4.5 and 4.6. The results for the non–degenerate case on meshes that do not align with the lines $y = \pm 1/2$ where derivatives of ν and α jump are similar to those shown in Tables 4.5 and 4.6.

Table 4.5. Non–symmetric stress $\mathcal{A}(G) = \nu G$.

h	$\|u-u_h\|_{L^2}$	$\|G-G_h\|_{L^2_{\mathcal{A}}}$	$\|G-G_h\|_{\mathbb{S}_h^*}$	$\|S-S_h\|_{L^2}$	$\|S-S_h\|_{\mathbb{H}(\mathrm{div})}$
1	42.06	247.83	34.52	319.79	4002.80
1/2	37.50	242.87	28.95	313.25	3987.85
1/4	24.48	190.47	15.92	202.70	3454.66
1/8	11.65	109.12	3.88	103.47	2091.00
1/16	5.92	51.68	1.17	46.40	1115.03
1/32	2.97	24.97	0.38	21.68	566.68
Rate	1.00	1.05	1.62	1.10	0.98
Norm	23.11	239.48		256.71	4001.66

Table 4.6. Symmetric stress $\mathcal{A}(G) = \nu(G + G^\top)$.

h	$\|u-u_h\|_{L^2}$	$\|G-G_h\|_{L^2_{\mathcal{A}}}$	$\|G-G_h\|_{\mathbb{S}_h^*}$	$\|S-S_h\|_{L^2}$	$\|S-S_h\|_{\mathbb{H}(\mathrm{div})}$
1	33.00	269.00	23.54	520.61	5931.59
1/2	34.36	273.38	24.99	520.41	5898.06
1/4	21.94	216.99	15.14	355.78	5147.94
1/8	11.95	128.14	4.69	190.98	3096.49
1/16	6.05	64.49	1.71	91.18	1649.59
1/32	2.99	31.65	0.55	41.89	838.11
Rate	1.02	1.03	1.65	1.12	0.98
Norm	23.11	270.91		431.71	5916.56

Errors and rates for degenerate case on rectangles, ν as in (4.1) and $\alpha = 1 - \nu$.

Appendix A. Proof of Lemma 3.2

Integration–by–parts shows

$$\begin{aligned}(G, T_h) &\equiv (-u, \mathrm{div}(T_h)) + \langle u, T_h n\rangle - (1/d)(\mathrm{div}(u), \mathrm{tr}(T_h))\\ &= (-u+u_p, \mathrm{div}(T_h)) + \langle u, T_h n\rangle - (1/d)(\mathrm{div}(u-u_p), \mathrm{tr}(T_h))\\ &\quad -(u_p, \mathrm{div}(T_h)) - (1/d)(\mathrm{div}(u_p), \mathrm{tr}(T_h))\\ &= (-u+u_p, \mathrm{div}(T_h)) - (1/d)(\mathrm{div}(u-u_p), \mathrm{tr}(T_h))\\ &\quad + \sum_{K\in\mathcal{T}_h} (\mathrm{dev}(\nabla u_p), T_h)_K - \sum_{e\in\mathcal{E}_h} \int_e ([u_p]\otimes n_e) : T_h.\end{aligned}$$

The estimate (3.8) is now obtained by rearranging terms and applying the Cauchy–Schwarz inequality.

Appendix B. Proof of Lemma 3.3

(1) The total number conditions is $(d+1)(d-1)\dim\mathcal{P}_k(\mathbb{R}^{d-1}) + (d^2-1)\dim\mathcal{P}_{k-1}(\mathbb{R}^d) = (d^2-1)(\binom{k+d-1}{d-1} + \binom{k+d-1}{d}) = \dim\mathrm{dev}(\mathcal{P}_k(K)^{d\times d})$. It then suffices to show that $H \in \mathrm{dev}(\mathcal{P}_k(K)^{d\times d})$ vanishes on the degrees of freedom if and only if $H \equiv 0$. We show this for the case $d = 3$; the two–dimensional case follows from similar arguments.

First we consider the case where $K = \hat{K}$ is the the reference tetrahedron. Label the faces so that $\hat{x}_k = 0$ on $\hat{f}_k$ and that $\hat{x}_1 + \hat{x}_2 + \hat{x}_3 = 1$ on $\hat{f}_4$. Explicit calculation shows that the condition

$(I - \hat{n} \otimes \hat{n})H\hat{n} = 0$ reduces to

$$H_{jk} = 0 \qquad \text{on } \hat{f}_k = 0 \qquad j \neq k,\ k = 1,2,3, \tag{B.1a}$$

$$3H_{11} + 2H_{12} + 2H_{13} - H_{21} - H_{23} - H_{31} - H_{32} = 0 \quad \text{on } \hat{f}_4, \tag{B.1b}$$

$$H_{21} + 2H_{22} + H_{23} - H_{31} - H_{32} + H_{11} = 0 \quad \text{on } \hat{f}_4. \tag{B.1c}$$

Since the moments of H on $\hat{K}$ vanish up to degree $(k-1)$, we find

$$0 = \int_{\hat{K}} \frac{\partial (H_{i,j}^2)}{\partial \hat{x}_k}\, d\hat{x} = \int_{\partial \hat{K}} H_{i,j}^2 \hat{n}^{(k)}\, d\hat{s} = -\int_{\hat{f}_k} H_{i,j}^2 d\hat{s} + \frac{1}{\sqrt{3}} \int_{\hat{f}_4} H_{i,j}^2\, d\hat{s}. \tag{B.2}$$

By (B.1a), $H_{jk} = 0$ vanishes on $\hat{f}_4$ for $k = 1,2,3$ and $j \neq k$. Therefore by (B.1b)–(B.1c), $H = 0$ on $\hat{f}_4$. Then using (B.2) and (B.1a), we conclude that H vanishes on $\partial\hat{K}$. Since the moments of H on $\hat{K}$ equal zero up to degree $(k-1)$, we conclude that $H \equiv 0$.

For a general element $K \in \mathcal{T}_h$, let $F_K : \hat{K} \to K$ denote an affine mapping with $F_K(\hat{x}) = B_K\hat{x} + b_K$, $B_K \in \mathbb{R}^{3\times 3}$ and $b_K \in \mathbb{R}^3$. Let $H \in \mathrm{dev}(\mathcal{P}_k(K)^{3\times 3})$ vanish at the degrees of freedom, and define $\hat{H} \in \mathrm{dev}(\mathcal{P}_k(\hat{K})^{3\times 3})$ via

$$B_K^{-T}\hat{H}(\hat{x})B_K^T = H(x), \qquad x = F_K(\hat{x}).$$

Since outward normal unit vectors and any tangential unit vectors satisfy the relations [27, p. 79]

$$n_i = \frac{B^{-T}\hat{n}_i}{\|B^{-T}\hat{n}_i\|}, \qquad \tau_i = \frac{B_K\hat{\tau}_i}{\|B_K\hat{\tau}_i\|},$$

there holds

$$\tau_i^T H n_i = \frac{\hat{\tau}_i^T \hat{H} \hat{n}_i}{\|B^{-T}\hat{n}_i\|\|B_K\hat{\tau}_i\|}.$$

Therefore, the moments of $(I - \hat{n}_i \otimes \hat{n}_i)\hat{H}\hat{n}_i$ vanish up to degree k on $\hat{f}_i$. It then follows that the moments of $\hat{H}$ vanish on $\hat{K}$ so $\hat{H} \equiv 0$ and hence $H \equiv 0$. This completes the proof of part (1).

(2) To prove part (2), first note that all of the off–diagonal entries of $\mathrm{dev}(T_h)$ (with $T_h \in \mathbb{RT}_k(K)^d$) vanish if and only if the moments of $(I - n_e \otimes n_e)T_h n_e$ vanish up to degree k on each $e \subset \partial K$. The result now follows upon noting that $(T_h, \mathrm{dev}(\kappa))_K = (\mathrm{dev}(T_h), \mathrm{dev}(\kappa))_K$.

Acknowledgements

The authors would like to thank the editor and four anonymous referees for their helpful comments that improved the presentation in the manuscript.

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