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Abstract. In this paper, a nonlinear control volume finite element (CVFE) scheme for a degenerate Keller–Segel model with anisotropic and heterogeneous diffusion tensors is proposed and analyzed. In this scheme, degrees of freedom are assigned to vertices of a primal triangular mesh, as in finite element methods. The diffusion term which involves an anisotropic and heterogeneous tensor is discretized on a dual mesh (Donald mesh) using the diffusion fluxes provided by the conforming finite element reconstruction on the primal mesh. The other terms are discretized using a nonclassical upwind finite volume scheme on the dual mesh. The scheme ensures the validity of the discrete maximum principle without any restriction on the transmissibility coefficients. The convergence of the scheme is proved under very general assumptions. Finally, some numerical experiments are carried out to prove the ability of the scheme to tackle degenerate anisotropic and heterogeneous diffusion problems over general meshes.

Math. classification. 65N08, 65N30.

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1. Introduction and model

In this paper, we are interested in degenerate nonlinear parabolic reaction–convection–diffusion systems modeling the chemotaxis process over general mesh, with anisotropic and heterogeneous diffusion tensors. From the numerical point of view, the convergence analysis of the finite volume scheme for this type of systems is carried out in [4] for the isotropic case (i.e. the diffusion tensor is considered to be proportional to the identity matrix) and under the "admissibility" assumption on the mesh used for the space discretization in the sense of satisfying the orthogonality condition (see e.g. [21]). Although its ability to ensure stability, the classical upwind finite volume method does not permit to handle anisotropic diffusion even if the mesh verifies the orthogonality condition. Various "multi-point" schemes, where the approximation of the flux through an edge involves several scalar unknowns, have been proposed for anisotropic diffusion problems, see for example [22, 18, 14, 2, 15] for a detailed review of modern finite volume methods for diffusion equations. However, nonlinear corrections have been proposed in [11] in order to enforce the monotony, but no complete convergence proof have been provided for such methods yet.

Let us introduce the chemotaxis model. For that, let Ω be a connected open bounded polygonal domain of $\mathbb{R}^2$, and $t_f > 0$ be a fixed time. The modified Keller–Segel system (e.g., see [26, 27]) modeling the chemotaxis process is given by the following set of equations

$$\begin{cases} \partial_t u - \operatorname{div}\left(\Lambda(\mathbf{x})a\left(u\right)\nabla u - \Lambda(\mathbf{x})\chi\left(u\right)\nabla v\right) = f\left(u\right) & \text{in } Q_{t_\mathrm{f}} = \Omega \times (0, t_\mathrm{f}), \\ \partial_t v - \operatorname{div}\left(D(\mathbf{x})\nabla v\right) = g\left(u, v\right) & \text{in } Q_{t_\mathrm{f}} = \Omega \times (0, t_\mathrm{f}). \end{cases} \tag{1.1}$$

The system is complemented with zeros-flux boundary conditions on $\Sigma_{t_\mathrm{f}} := \partial\Omega \times (0, t_\mathrm{f})$ given by

$$\left(\Lambda(\mathbf{x})a\left(u\right)\nabla u - \Lambda(\mathbf{x})\chi\left(u\right)\nabla v\right)\cdot \mathbf{n} = 0, \qquad D(\mathbf{x})\nabla v \cdot \mathbf{n} = 0, \tag{1.2}$$

and the initial conditions on Ω:

$$u\left(\mathbf{x}, 0\right) = u_0(\mathbf{x}), \quad v\left(\mathbf{x}, 0\right) = v_0(\mathbf{x}). \tag{1.3}$$

In the above model, the density of the cell-population and the chemoattractant concentration are represented by $u = u\left(\mathbf{x}, t\right)$ and $v = v\left(\mathbf{x}, t\right)$ respectively. Next, $a(u)$ is a density-dependent diffusion coefficient, and $\Lambda(\mathbf{x})$ is the diffusion tensor in a heterogeneous medium. Furthermore, the function χ is the chemoattractant sensitivity, and $D(\mathbf{x})$ is the diffusion tensor for v. The function f describes the cell density proliferation and the cell density death. The function g describes the production and the degradation of the chemoattractant concentration; for simplicity, we assume that it is a linear function given by

$$g\left(u, v\right) = \alpha u - \beta v, \qquad \alpha, \beta \geq 0. \tag{1.4}$$

α and β represent respectively the production and the degradation rate of the chemical concentration.

Let us state the main assumptions made about system (1.1)–(1.3):

(A1) The cell-density diffusion coefficient $a : [0, 1] \longrightarrow \mathbb{R}^+$ is a continuous function such that, $a(0) = a(1) = 0$, and $a(u) > 0$ for $0 < u < 1$.

(A2) The chemosensitivity $\chi : [0, 1] \longrightarrow \mathbb{R}^+$ is a continuous function such that, $\chi(0) = \chi(1) = 0$. Furthermore, we assume that there exists a function $\mu \in \mathcal{C}\left([0, 1]; \mathbb{R}^+\right)$, such that $\mu\left(u\right) = \dfrac{\chi\left(u\right)}{a\left(u\right)}$ for all $u \in (0, 1)$ and $\mu(0) = \mu(1) = 0$.

(A3) The diffusion tensors Λ and D are two bounded, uniformly positive symmetric tensors on Ω, that is: $\forall \mathbf{w} \neq 0, 0 < T_- \left|\mathbf{w}\right|^2 \leq \langle T(\mathbf{x})\mathbf{w}, \mathbf{w}\rangle \leq T_+ \left|\mathbf{w}\right|^2 < \infty, \quad T = \Lambda$ or D.

(A4) The cell density proliferation f is a continuous function such that $f\left(0\right) \geq 0$ and $f\left(1\right) \leq 0$.

(A5) The initial function u_0 and v_0 are two functions in $L^2\left(\Omega\right)$ such that, $0 \leq u_0 \leq 1$ and $v_0 \geq 0$.

In the sequel, we use the Lipschitz continuous nondecreasing function $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\xi\left(u\right) := \int_0^u \sqrt{a\left(s\right)}\,\mathrm{d}s, \qquad \forall u \in \mathbb{R}. \tag{1.5}$$

We recall the definition of a weak solution of system (1.1)–(1.3).

Definition 1.1 (weak solution)**.** Under the assumptions (A1)–(A5), we say that the couple of measurable functions (u, v) is a weak solution of system (1.1)–(1.3) if

$$\begin{aligned} &0 \leq u\left(\mathbf{x}, t\right) \leq 1, \qquad 0 \leq v\left(\mathbf{x}, t\right) \quad \text{for a.e. in } Q_{t_\mathrm{f}}, \\ &\xi\left(u\right) \in L^2\left(0, t_\mathrm{f}; H^1\left(\Omega\right)\right), \\ &v \in L^\infty\left(Q_{t_\mathrm{f}}\right) \cap L^2\left(0, t_\mathrm{f}; H^1\left(\Omega\right)\right), \end{aligned}$$

and for all $\varphi, \psi \in \mathcal{D}\left(\overline{\Omega} \times [0, t_{\mathrm{f}})\right)$, one has

$$-\int_{\Omega} u_0(\mathbf{x})\varphi(\mathbf{x}, 0)\,\mathrm{d}\mathbf{x} - \iint_{Q_{t_{\mathrm{f}}}} u\,\partial_t\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \iint_{Q_{t_{\mathrm{f}}}} \sqrt{a(u)}\Lambda(\mathbf{x})\nabla\xi(u)\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$- \iint_{Q_{t_{\mathrm{f}}}} \Lambda(\mathbf{x})\chi(u)\nabla v\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \iint_{Q_{t_{\mathrm{f}}}} f(u)\,\varphi(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{1.6}$$
$$-\int_{\Omega} v_0(\mathbf{x})\psi(\mathbf{x}, 0)\,\mathrm{d}\mathbf{x} - \iint_{Q_{t_{\mathrm{f}}}} v\,\partial_t\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \iint_{Q_{t_{\mathrm{f}}}} D(\mathbf{x})\nabla v\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \iint_{Q_{t_{\mathrm{f}}}} g(u, v)\,\psi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{1.7}$$

A standard weak formulation uses the Kirchhoff transform $\kappa(u)$ as a primitive of the function $a(u)$. According to [8, 9], we approximate the degenerate diffusion term in its original form in (1.1). Next, we use the specific form of the chemoattractant function to propose a new scheme preserving the positivity of solutions and convergent.

Schemes with mixed conforming piecewise linear finite elements on triangles for the diffusion term and finite volume on dual elements were proposed and analyzed in [7, 13, 1] for fluid mechanics equations, and in [25] for a degenerate nonlinear chemotaxis model. The convergence analysis for these schemes is carried out for the case of anisotropic and heterogeneous diffusion problems under an essential assumption that all the transmissibility coefficients are nonnegative. However, there is no sufficient conditions for nonnegativity of transmissibility coefficients and therefore the schemes do not permit to tackle general anisotropic diffusion problems. Nevertheless, in [12] the authors propose a combined nonconforming finite elements finite volumes scheme for which they add a monotone regularization permitting positiveness of discrete solution; the convergence of the scheme, introduced in [11], is ensured under a numerical condition depending on the mesh size and on the discrete solutions.

Recently, Cancès and Guichard proposed and analyzed in [9] a nonlinear *Control Volume Finite Element* (CVFE) scheme for solving degenerate anisotropic parabolic diffusion equations modeling flows in porous media. The convergence analysis is carried out without any restriction on the transmissibility coefficients, and the efficiency of the scheme is tested using anisotropic diffusion tensors over an unstructured mesh.
Our aim is to elaborate a general approach, inspired from [9] and [25], to approximate a nonlinear degenerate parabolic system modeling the chemotaxis process over general mesh, with anisotropic and heterogeneous diffusion tensors. Especially, the diffusion terms are discretized by means of a conforming piecewise linear finite element method on a primal triangular mesh and using the Godunov scheme to approximate the diffusion fluxes provided by the conforming finite element reconstruction. The others terms are discretized by means of a nonclassical upwind finite volume method on a dual mesh (Donald mesh or Median dual mesh).

The rest of this paper is organized as follows. In section 2, we define a primal triangular mesh and its corresponding Donald dual mesh, next, we define standard $\mathbb{P}_1$ finite element and finite volume reconstructions. Then, we introduce the nonlinear CVFE scheme and specify the discretization of the degenerate diffusion and convection terms. In Section 3, we prove the existence of a discrete solution to the CVFE scheme based on the establishment of *a priori* estimates on the discrete solution as well as the discrete maximum principle. In Section 4, we give estimates on differences of time and space translates for the approximate solutions. In Section 5, using the Kolmogorov relative compactness criterion, we prove the convergence of a subsequent of discrete solutions to the weak solution (Definition 1.1). Finally, some numerical simulations are carried out, in Section 6, to show the effectiveness of the scheme to tackle degenerate anisotropic and heterogeneous diffusion problems over general unstructured mesh.

2. The numerical scheme and main result

In this section, we describe the space and time discretizations of Q_{t_f}, define the approximate spaces, introduce useful properties on discrete H^1-norms stemming from finite elements discretizations as well as the *nonlinear CVFE* scheme, and state the main result.

2.1. Space-time discretization and notations

2.1.1. *Space discretizations of* Ω.

In order to discretize problem (1.1)–(1.3), we perform a finite element triangulation $\mathcal{T}$ of the polygonal domain Ω, consisting of open bounded triangles such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}} \overline{T}$ and such that for all $T, T' \in \mathcal{T}$, $\overline{T} \cap \overline{T'}$ is either an empty set or a common vertex or edge of T and T'. We denote by $\mathcal{V}$ the set of vertices of the discretization $\mathcal{T}$, located at positions $(\mathbf{x}_K)_{K\in\mathcal{V}}$, and by $\mathcal{E}$ the set of edges of $\mathcal{T}$ joining two vertices of $\mathcal{V}$. The edge joining two vertices K and L is denoted by σ_{KL}.

For a given triangle $T \in \mathcal{T}$, we denote by $\mathbf{x}_T$ the centre of gravity of T, by $\mathcal{E}_T$ the set of the edges of T, by h_T the diameter of T, and by ρ_T the diameter of the largest ball inscribed in the triangle T. We denote by h the size of the triangulation $\mathcal{T}$ defined by $h := \max_{T\in\mathcal{T}} h_T$ and and by $\theta_{\mathcal{T}}$ the shape regularity of the triangulation $\mathcal{T}$, defined by $\theta_{\mathcal{T}} := \max_{T\in\mathcal{T}} h_T/\rho_T$.

For $K \in \mathcal{V}$, we denote by $\mathcal{E}_K$ the set of the edges having K as an extremity, and by $\mathcal{T}_K$ the subset of $\mathcal{T}$ including the triangles having K as a vertex. We also define a barycentric dual mesh $\mathcal{M}$ (known as

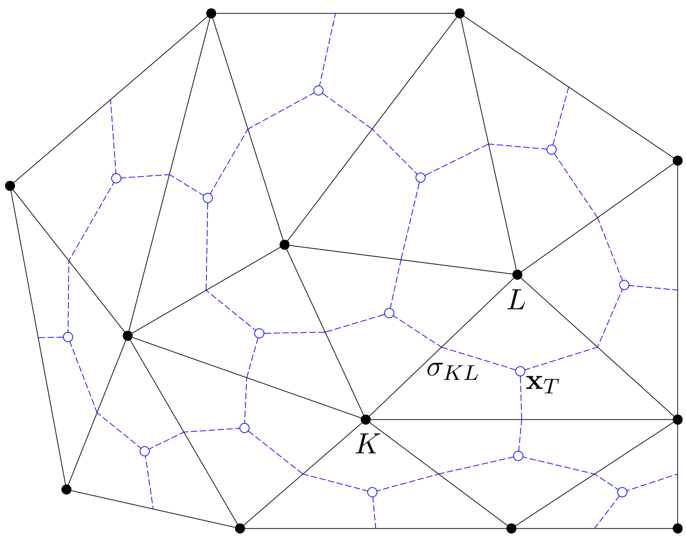

Figure 2.1. Triangular mesh $\mathcal{T}$ and Donald dual mesh $\mathcal{M}$: dual volumes, vertices, interfaces.

Donald dual or Median dual mesh) generated by the triangulation mesh $\mathcal{T}$. There is one dual element ω_K associated with each vertex $K \in \mathcal{V}$. We construct it around the vertex K by connecting the barycenter $\mathbf{x}_T$ of each surrounding triangle $T \in \mathcal{T}_K$ with the barycenters $\mathbf{x}_\sigma$ of the edges $\sigma \in \mathcal{E}_K$. We refer to Fig. 2.1 for an illustration of the primal and the barycentric dual mesh in a two-dimensional space. Note that $\overline{\Omega} = \bigcup_{K\in\mathcal{V}} \overline{\omega}_K$. The 2-dimensional Lebesgue measure of ω_K is denoted by m_K.

2.2. Discrete finite elements space $\mathcal{H}_{\mathcal{T}}$, control volumes space $\mathcal{X}_{\mathcal{M}}$.

We define two discrete functional spaces associated with each mesh of the above meshes. The first one, denoted by $\mathcal{H}_{\mathcal{T}}$, is the usual $\mathbb{P}_1$-finite element space corresponding to the triangular mesh $\mathcal{T}$, consisting of piecewise affine finite elements.

$$\mathcal{H}_{\mathcal{T}} := \left\{\varphi \in \mathcal{C}^0\left(\overline{\Omega}\right);\, \varphi_{|_T} \in \mathbb{P}_1(\mathbb{R}),\ \forall T \in \mathcal{T}\right\} \subset H^1(\Omega).$$

The canonical basis of $\mathcal{H}_{\mathcal{T}}$ is spanned by the shape functions $(\varphi_K)_{K\in\mathcal{V}}$, such that

$$\varphi_K(\mathbf{x}_K) = 1, \quad \varphi_K(\mathbf{x}_L) = 0 \text{ if } L \neq K, \qquad \forall K \in \mathcal{V}.$$

On the other hand, we denote by $\mathcal{X}_{\mathcal{M}}$ the discrete control volumes space consisting of piecewise constant functions on the dual mesh $\mathcal{M}$.

$$\mathcal{X}_{\mathcal{M}} = \{\varphi : \Omega \longrightarrow \overline{\mathbb{R}} \text{ measurable; } \varphi_{|_{\omega_K}} \in \overline{\mathbb{R}} \text{ is constant}, \forall K \in \mathcal{V}\}.$$

Given a vector $(u_K)_{K\in\mathcal{V}} \in \mathbb{R}^{\#\mathcal{V}}$(resp. $(v_K)_{K\in\mathcal{V}} \in \mathbb{R}^{\#\mathcal{V}}$), there exists a unique function $u_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}$ (resp. $v_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}$) and a unique $u_{\mathcal{M}} \in \mathcal{X}_{\mathcal{M}}$ (resp. $v_{\mathcal{M}} \in \mathcal{X}_{\mathcal{M}}$) such that

$$\begin{aligned} u_{\mathcal{T}}(\mathbf{x}_K) &= u_{\mathcal{M}}(\mathbf{x}_K) = u_K, && \forall K \in \mathcal{V}, \\ v_{\mathcal{T}}(\mathbf{x}_K) &= v_{\mathcal{M}}(\mathbf{x}_K) = v_K, && \forall K \in \mathcal{V}. \end{aligned} \tag{2.1}$$

For all $(K, L) \in \mathcal{V}^2$, we define the transmissibility coefficient T_{KL} by

$$T_{KL} = -\int_\Omega T(\mathbf{x})\nabla\varphi_K(\mathbf{x}) \cdot \nabla\varphi_L(\mathbf{x})\mathrm{d}\mathbf{x} = T_{LK}, \qquad T = \Lambda \text{ or } D. \tag{2.2}$$

We have $T_{KK} = -\sum_{L\neq K} T_{KL}$, since $\sum_{K\in\mathcal{V}} \nabla\varphi_K = 0$. As a consequence, one has

$$\int_\Omega T(\mathbf{x})\nabla u_{\mathcal{T}} \cdot \nabla v_{\mathcal{T}}\mathrm{d}\mathbf{x} = \sum_{\sigma_{KL}\in\mathcal{E}} T_{KL}(u_K - u_L)(v_K - v_L), \quad T = \Lambda \text{ or } D.$$

2.3. Time discretization of $(0, t_{\mathrm{f}})$.

For the time discretization of the interval $(0, t_{\mathrm{f}})$, we consider a uniform time discretization, and we do not impose any restriction on the time step. In addition, we assume that the spatial meshes do not change with the time step. We note that all the results presented in this paper can be extended to the case of general time discretization.
Let N be a nonnegative integer, we define the uniform time step $\Delta t = t_{\mathrm{f}}/(N+1)$, and $t^n = n\Delta t$ for all $n \in \{0, \ldots, N+1\}$, so that $t^0 = 0$, and $t^{N+1} = t_{\mathrm{f}}$.

2.4. Space-time discretization of $Q_{t_{\mathrm{f}}}$.

Here, we define the space and time discrete spaces $\mathcal{H}_{\mathcal{T},\Delta t}$ and $\mathcal{X}_{\mathcal{M},\Delta t}$ as the set of piecewise constant functions in time with values in $\mathcal{H}_{\mathcal{T}}$ and $\mathcal{X}_{\mathcal{T}}$ respectively.

$$\begin{aligned} \mathcal{H}_{\mathcal{T},\Delta t} &= \{\varphi \in L^2\left(0, t_{\mathrm{f}}; H^1(\Omega)\right), \varphi(\mathbf{x}, t) = \varphi\left(\mathbf{x}, t^{n+1}\right) \in \mathcal{H}_{\mathcal{T}}, \quad \forall t \in (t^n, t^{n+1}]\}, \\ \mathcal{X}_{\mathcal{M},\Delta t} &= \{\varphi : Q_{t_{\mathrm{f}}} \longrightarrow \overline{\mathbb{R}} \text{ measurable}, \varphi(\mathbf{x}, t) = \varphi\left(\mathbf{x}, t^{n+1}\right) \in \mathcal{X}_{\mathcal{M}}, \quad \forall t \in (t^n, t^{n+1}]\}. \end{aligned}$$

For a given $(u_K^n)_{n\in\{0,\cdots,N+1\},K\in\mathcal{V}} \in \mathbb{R}^{(N+2)\#\mathcal{V}}$ (resp. $(v_K^n)_{n\in\{0,\cdots,N+1\},K\in\mathcal{V}}$), there exists a unique function $u_{\mathcal{T},\Delta t} \in \mathcal{H}_{\mathcal{T},\Delta t}$ (resp. $v_{\mathcal{T},\Delta t} \in \mathcal{H}_{\mathcal{T},\Delta t}$) and a unique $u_{\mathcal{M},\Delta t} \in \mathcal{X}_{\mathcal{M},\Delta t}$ (resp. $v_{\mathcal{M},\Delta t} \in \mathcal{X}_{\mathcal{M},\Delta t}$)

such that

$$\begin{aligned} u_{\mathcal{T},\Delta t}\left(\mathbf{x}_K,t\right) &= u_{\mathcal{M},\Delta t}\left(\mathbf{x}_K,t\right) = u_K^{n+1}, \qquad \forall K \in \mathcal{V},\, \forall t \in (t^n, t^{n+1}], \\ v_{\mathcal{T},\Delta t}\left(\mathbf{x}_K,t\right) &= v_{\mathcal{M},\Delta t}\left(\mathbf{x}_K,t\right) = v_K^{n+1}, \qquad \forall K \in \mathcal{V},\, \forall t \in (t^n, t^{n+1}]. \end{aligned} \tag{2.3}$$

2.5. **The nonlinear CVFE scheme**

The discretizations of the initial data u_K^0 and v_K^0, $K \in \mathcal{V}$ are defined by

$$u_{\mathcal{M}}^0\left(\mathbf{x}\right) = u_K^0 = \frac{1}{m_K}\int_{\omega_K} u_0\left(\mathbf{y}\right)\,\mathrm{d}\mathbf{y}, \qquad \forall \mathbf{x} \in \omega_K, \tag{2.4}$$

$$v_{\mathcal{M}}^0\left(\mathbf{x}\right) = v_K^0 = \frac{1}{m_K}\int_{\omega_K} v_0\left(\mathbf{y}\right)\,\mathrm{d}\mathbf{y}, \qquad \forall \mathbf{x} \in \omega_K, \tag{2.5}$$

2.5.1. *Discretization of the first equation of system* (1.1)

For all $K \in \mathcal{V}$, and $n \in \{0,\dots,N\}$, we define the discretization of the diffusion term by

$$\sum_{\sigma_{KL}\in\mathcal{E}_K} a_{KL}^{n+1}\Lambda_{KL}\left(u_K^{n+1} - u_L^{n+1}\right),$$

where,

$$a_{KL}^{n+1} = \begin{cases} \max\limits_{u\in I_{KL}^{n+1}} a\left(u\right) & \text{if } \Lambda_{KL} \geq 0, \\ \min\limits_{u\in I_{KL}^{n+1}} a\left(u\right) & \text{if } \Lambda_{KL} \leq 0, \end{cases} \tag{2.6}$$

and I_{KL}^{n+1} denotes the interval defined by

$$I_{KL}^{n+1} = [\min(u_K^{n+1}, u_L^{n+1}), \max(u_K^{n+1}, u_L^{n+1})]$$

Let us focus on the discretization of the convection term, and recall that the function $\chi\left(u\right)$ is defined to be the product of the continuous functions $\mu\left(u\right)$ and $a\left(u\right)$. To handle the discretization of the convection term in order to obtain a robust and stable scheme, we perform a nonclassical upwind finite volume scheme which consists of considering an upwind scheme for the function $\mu\left(u\right)$ according to the discrete gradient of v, and an upwind finite volume scheme for the function $a\left(u\right)$ with respect to u. These choices of discretization are crucial to ensure the discrete maximum principle as well as the energy estimates on the approximate solutions.

Definition 2.1. Consider system (1.1) and the notations given in §2.2. We say that the function $\mu_{KL}^{n+1} = z\left(u_K^{n+1}, u_L^{n+1}\right)$ is an approximation of the function $\mu\left(u\right)$ on the interfaces of ω_K with respect to the discrete gradient of v if it is nonincreasing (resp. nondecreasing) with respect to the first variable u_K^{n+1} and nondecreasing (resp. nonincreasing) with respect to the other variable u_L^{n+1} when $\Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) \geq 0$ (resp. $\Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) \leq 0$). Furthermore, we have $z\left(u_K^{n+1}, u_K^{n+1}\right) = \mu\left(u_K^{n+1}\right)$.

We give here two examples on the construction of μ_{KL}^{n+1}. The first example consists of taking the Engquist-Osher scheme and the second example consists of taking the Godunov scheme (see e.g. [24, 28]).

Engquist-Osher scheme

- $\mu_{KL}^{n+1} = \begin{cases} \mu_{\downarrow}\left(u_K^{n+1}\right) + \mu_{\uparrow}\left(u_L^{n+1}\right), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) \geq 0, \\ \mu_{\uparrow}\left(u_K^{n+1}\right) + \mu_{\downarrow}\left(u_L^{n+1}\right), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) < 0. \end{cases}$

  The functions $\mu_{\uparrow}$ and $\mu_{\downarrow}$ are given by

$$\mu_{\uparrow}(z) := \int_0^z \left(\mu'(s)\right)^+ \, \mathrm{d}s, \qquad \mu_{\downarrow}(z) := -\int_0^z \left(\mu'(s)\right)^- \, \mathrm{d}s.$$

  Herein, $s^+ = \max(s, 0)$ and $s^- = \max(-s, 0)$.

Godunov scheme

- $\mu_{KL}^{n+1} = \begin{cases} \max\limits_{[u_K^{n+1}, u_L^{n+1}]} \mu(u), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) \geq 0, \text{ and } u_K^{n+1} \leq u_L^{n+1}, \\ \min\limits_{[u_L^{n+1}, u_K^{n+1}]} \mu(u), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) \geq 0, \text{ and } u_K^{n+1} > u_L^{n+1}, \\ \max\limits_{[u_L^{n+1}, u_K^{n+1}]} \mu(u), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) < 0, \text{ and } u_K^{n+1} > u_L^{n+1}, \\ \min\limits_{[u_K^{n+1}, u_L^{n+1}]} \mu(u), & \text{if } \Lambda_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) < 0, \text{ and } u_K^{n+1} \leq u_L^{n+1}. \end{cases}$

We are now in a position to introduce what we call *nonlinear control volume finite element (CVFE) scheme.* For all $K \in \mathcal{V}$, and all $n \in \{0, \ldots, N\}$,

$$\frac{u_K^{n+1} - u_K^n}{\Delta t} m_K + \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right) - \sum_{\sigma_{KL} \in \mathcal{E}_K} \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1}\left(v_K^{n+1} - v_L^{n+1}\right) = f\left(u_K^{n+1}\right) m_K, \tag{2.7}$$

where, the transmissibility coefficients Λ_{KL} and D_{KL} are given in equality (2.2).

2.5.2. *Discretization of the second equation of system* (1.1)–(1.3)

Here, we focus on the discretization of the second equation of system (1.1)–(1.3). We note that a classical discretization of this equation is given by the following form

$$m_K \frac{v_K^{n+1} - v_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} D_{KL}\left(v_K^{n+1} - v_L^{n+1}\right) = m_K\left(\alpha u_K^n - \beta v_K^{n+1}\right). \tag{2.8}$$

However, this discretization does not guaranty the positivity of the discrete solutions without any restriction on the transmissibility coefficients, for instance, one can get the discrete maximum principle by assuming that all the transmissibility coefficients D_{KL} are nonnegative (see [25]).
Here, we propose a numerical discretization in order to ensure the discrete maximum principle without any restriction on the transmissibility coefficients. To do this, we introduce the following set of

functions: $\eta(v)$, $p(v)$, $\Gamma(v)$ and $\phi(v)$ defined by

$$\eta(v) = \max(0, \min(v,1)), \tag{2.9}$$

$$p(v) = \int_1^v \frac{1}{\eta(s)}\,\mathrm{d}s = \begin{cases} \ln(v) & \text{if } v \in (0,1), \\ v-1 & \text{if } v \geq 1, \end{cases} \tag{2.10}$$

$$\Gamma(v) = \int_1^v p(s)\,\mathrm{d}s = \begin{cases} v\ln(v) - v + 1 & \text{if } v \in [0,1), \\ \frac{(v-1)^2}{2} & \text{if } v \geq 1, \end{cases} \tag{2.11}$$

$$\phi(v) = \int_0^v \frac{1}{\sqrt{\eta(s)}}\,\mathrm{d}s = \begin{cases} 2\sqrt{v} & \text{if } v \in [0,1), \\ v+1 & \text{if } v \geq 1. \end{cases} \tag{2.12}$$

In the sequel, we adopt the convention

$$\eta(v)p(v) = 0 \quad \text{if } v \leq 0. \tag{2.13}$$

We give the discretization of the second equation of system (1.1)–(1.3); specifically, we have

$$m_K \frac{v_K^{n+1} - v_K^n}{\Delta t} + \sum_{\sigma_{KL} \in \mathcal{E}_K} D_{KL} \eta_{KL}^{n+1} \left(p\left(v_K^{n+1}\right) - p\left(v_L^{n+1}\right)\right) = m_K \left(\alpha u_K^n - \beta v_K^{n+1}\right), \tag{2.14}$$

where, denoting by $J_{KL}^{n+1} = [\min(v_K^{n+1}, v_L^{n+1}), \max(v_K^{n+1}, v_L^{n+1})]$, we have set

$$\eta_{KL}^{n+1} = \begin{cases} \max_{s \in J_{KL}^{n+1}} \eta(s) & \text{if } D_{KL} \geq 0, \\ \min_{s \in J_{KL}^{n+1}} \eta(s) & \text{if } D_{KL} < 0. \end{cases} \tag{2.15}$$

Note that because of the use of the function p in the scheme, the scheme (2.14) only makes sense if

$$v_K^{n+1} > 0 \qquad \forall K \in \mathcal{V},\ \forall n \geq 0. \tag{2.16}$$

This will be assumed in the *a priori* estimates and rigorously proved later on (cf. Lemma 3.11).

We can show that the scheme (2.7)–(2.14), whose construction is based on finite elements for the diffusion term and a nonclassical upwind finite volume for the convection term, can be interpreted as a finite volume scheme. Indeed, denoting by

$$\begin{aligned} F_{KL}^{n+1} &= \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right) - \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1} \left(v_K^{n+1} - v_L^{n+1}\right), \\ \Phi_{KL}^{n+1} &= D_{KL} \eta_{KL}^{n+1} \left(p\left(v_K^{n+1}\right) - p\left(v_L^{n+1}\right)\right). \end{aligned}$$

Then the scheme (2.7)–(2.14) rewrites

$$\begin{cases} F_{KL}^{n+1} + F_{LK}^{n+1} = 0 = \Phi_{KL}^{n+1} + \Phi_{LK}^{n+1}, & \text{for all } \sigma_{KL} \in \mathcal{E}, \\ m_K \dfrac{u_K^{n+1} - u_K^n}{\Delta t} + \displaystyle\sum_{\sigma_{KL} \in \mathcal{E}_K} F_{KL}^{n+1} = f\left(u_K^{n+1}\right) m_K, & \text{for all } K \in \mathcal{V}, \\ m_K \dfrac{v_K^{n+1} - v_K^n}{\Delta t} + \displaystyle\sum_{\sigma_{KL} \in \mathcal{E}_K} \Phi_{KL}^{n+1} = g\left(u_K^n, v_K^{n+1}\right) m_K, & \text{for all } K \in \mathcal{V}. \end{cases}$$

2.6. **Main result**

Let $(\mathcal{T}_m)_{m \geq 1}$ be a sequence of triangulations of Ω such that

$$h_m = \max_{T \in \mathcal{T}_m} \operatorname{diam}(T) \to 0 \text{ as } m \to \infty.$$

We assume that the sequence of triangulations has a bounded regularity, i.e., there exists a constant $\theta > 0$ such that

$$\theta_{\mathcal{T}_m} \leq \theta, \quad \forall m \geq 1.$$

As before, a sequence of dual meshes $(\mathcal{M}_m)_{m\geq 1}$ is given.

Let $(N_m)_m$ be an increasing sequence of integers, then we define the corresponding sequence of time steps $(\Delta t_m)_m$ such that $\Delta t_m \to 0$ as $m \to \infty$. The intention of this paper is to prove the following main result.

Theorem 2.2. *Let $(u_{\mathcal{M}_m,\Delta t_m}, v_{\mathcal{M}_m,\Delta t_m})_m$ be a sequence of solutions to the scheme* (2.7)–(2.14)*, such that $0 \leq u_{\mathcal{M}_m,\Delta t_m} \leq 1$ and $0 \leq v_{\mathcal{M}_m,\Delta t_m}$ for almost everywhere in Q_{t_f}, then*

$$u_{\mathcal{M}_m,\Delta t_m} \to u \text{ and } v_{\mathcal{M}_m,\Delta t_m} \to v \quad \text{a.e. in } Q_{t_\mathrm{f}} \text{ as } m \to \infty,$$

where the couple (u, v) is a weak solution to the system (1.1)–(1.3) *in the sense of Definition 1.1.*

3. Discrete properties, a priori estimates and existence of a discrete solution

In this section, we first bring up some technical lemmas presented by Cancès and Guichard [9], that we reproduce here for clarity. Then, we establish the a *priori* estimates necessary to prove later the existence of a solution to the discrete problem.

Lemma 3.1. *Let $\left(u_K^{n+1}\right)_{K,n} \in \mathbb{R}^{(N+1)\#\mathcal{V}}$ (resp. $\left(v_K^{n+1}\right)_{K,n} \in \mathbb{R}^{(N+1)\#\mathcal{V}}$), then denoting by $\xi_{\mathcal{T},\Delta t}$ (resp. $\phi_{\mathcal{T},\Delta t}$) the unique function of $\mathcal{H}_{\mathcal{T},\Delta t}$ with nodal values $\left(\xi\left(u_K^{n+1}\right)\right) \in \mathbb{R}^{(N+1)\#\mathcal{V}}$ (resp. $\left(\phi\left(v_K^{n+1}\right)\right) \in \mathbb{R}^{(N+1)\#\mathcal{V}}$), one has*

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 \\ \geq \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} \left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2 = \iint_{Q_{t_\mathrm{f}}} \Lambda\nabla\xi_{\mathcal{T},\Delta t} \cdot \nabla\xi_{\mathcal{T},\Delta t} \mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL} \eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2 \\ \geq \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL} \left(\phi\left(v_K^{n+1}\right) - \phi\left(v_L^{n+1}\right)\right)^2 = \iint_{Q_{t_\mathrm{f}}} D\nabla\phi_{\mathcal{T},\Delta t} \cdot \nabla\phi_{\mathcal{T},\Delta t} \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned} \tag{3.2}$$

Proof. We refer to [9, Lemma 3.1], for the proof of this lemma. ■

Let $T \in \mathcal{T}$, and let $(K, L) \in \mathcal{V}^2$, we denote by

$$\lambda_{KL}^T := -\int_T \Lambda\nabla\varphi_K \cdot \nabla\varphi_L \mathrm{d}\mathbf{x} = \lambda_{LK}^T.$$

$$\delta_{KL}^T := -\int_T D\nabla\varphi_K \cdot \nabla\varphi_L \mathrm{d}\mathbf{x} = \delta_{LK}^T.$$

As a consequence, $\Lambda_{KL} = \sum_{T\in\mathcal{T}} \lambda_{KL}^T$ and $D_{KL} = \sum_{T\in\mathcal{T}} \delta_{KL}^T$ for all $\sigma_{KL} \in \mathcal{E}$.

Lemma 3.2. *Let $\Psi_{\mathcal{T}} = \sum_{K\in\mathcal{V}} \psi_K \varphi_K \in \mathcal{H}_{\mathcal{T}}$, then there exists a quantity C_0 depending only on Λ, D, and $\theta_{\mathcal{T}}$ such that*

$$\sum_{\sigma_{KL}\in\mathcal{E}} \sum_{T\in\mathcal{T}} \left|\lambda_{KL}^T\right| (\psi_K - \psi_L)^2 \le C_0 \int_\Omega \Lambda\nabla\Psi_{\mathcal{T}} \cdot \nabla\Psi_{\mathcal{T}} \mathrm{d}\mathbf{x}, \tag{3.3}$$

and

$$\sum_{\sigma_{KL}\in\mathcal{E}} \sum_{T\in\mathcal{T}} \left|\delta_{KL}^T\right| (\psi_K - \psi_L)^2 \le C_0 \int_\Omega D\nabla\Psi_{\mathcal{T}} \cdot \nabla\Psi_{\mathcal{T}} \mathrm{d}\mathbf{x}. \tag{3.4}$$

Proof. We refer to [9, Lemma 3.2] for the proof of this lemma. ∎

Lemma 3.3. *There exists a quantity C_1 depending only on Λ, D, and $\theta_{\mathcal{T}}$ such that*

$$\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}|\, a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 \le C_1 \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2. \tag{3.5}$$

and

$$\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |D_{KL}|\, \eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2 \le C_1 \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL} \eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2. \tag{3.6}$$

Proof. We denote by $\mathcal{E}^- := \{\sigma_{KL} \in \mathcal{E};\, \Lambda_{KL} < 0\}$, then since $|\mathbf{x}| = \mathbf{x} + 2\mathbf{x}^-$, $\mathbf{x}^- = \max(-\mathbf{x}, 0)$, one has

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}|\, a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 &= \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 \\ &\quad + 2\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}^-} |\Lambda_{KL}|\, a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2.\end{aligned}$$

Now, from the definition (2.6) of a_{KL}^{n+1}, there exists $c \in \overset{\circ}{I}{}_{KL}^{n+1}$ such that

$$\left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2 = a(c)\left(u_K^{n+1} - u_L^{n+1}\right)^2 \ge a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right)^2, \quad \forall \sigma_{KL} \in \mathcal{E}^-$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}|\, a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 &\le \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2 \\ &\quad + 2\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}| \left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2.\end{aligned} \tag{3.7}$$

Lemma 3.2 ensures the existence of a quantity $C_0 > 0\,(= C_0(\Lambda, \theta_{\mathcal{T}}))$ such that

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}| \left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2 &\le \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \sum_{T\in\mathcal{T}} |\lambda_{KL}| \left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2 \\ &\le C_0 \int_{Q_t} \Lambda\nabla\xi_{\mathcal{T},\Delta t} \cdot \nabla\xi_{\mathcal{T},\Delta t} \mathrm{d}\mathbf{x}\, \mathrm{d}t,\end{aligned}$$

and from Lemma 3.1, we deduce that

$$\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |\Lambda_{KL}| \left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)^2 \le C_0 \sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1}\right)^2. \tag{3.8}$$

Plugging estimate (3.8) into estimate (3.7), then estimate (3.5) holds with $C_1 = 1 + 2C_0$. The proof of estimate (3.6) is similar. ■

3.1. **Discrete maximum principle**

Lemma 3.4. *Let* $\left(u_K^{n+1}, v_K^{n+1}\right)_{K\in\mathcal{V},\, n\in\{0,\ldots,N\}}$ *be a solution to the CVFE scheme* (2.7)–(2.14)*. Then, for all* $K \in \mathcal{V}$*, and all* $n \in \{0,\ldots,N+1\}$*, we have* $0 \le u_K^n \le 1$*.*

Proof. We show this property using an induction on n. The property is true for $n = 0$ thanks to the definitions (2.4) and (2.5) of u_K^0 and v_K^0 and to the assumptions on u_0 and v_0. Now, assume that the claim is true up to time step n. Consider a dual control volume ω_K such that $u_K^{n+1} = \min_{L\in\mathcal{V}}\{u_L^{n+1}\}$, we want to show that $u_K^{n+1} \ge 0$ i.e. $\left(u_K^{n+1}\right)^- = 0$. Multiplying equation (2.7) by $-\left(u_K^{n+1}\right)^-$, one has

$$\begin{aligned}-m_K\frac{u_K^{n+1}-u_K^n}{\Delta t}\left(u_K^{n+1}\right)^- &- \sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1}-u_L^{n+1}\right)\left(u_K^{n+1}\right)^- \\ &+ \sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1}-v_L^{n+1}\right)\left(u_K^{n+1}\right)^- = -m_K f\left(u_K^{n+1}\right)\left(u_K^{n+1}\right)^- \le 0,\end{aligned} \tag{3.9}$$

to which, we have used the extension by $f(0) \ge 0$ (see assumption (A4)) of the continuous function f for $u \le 0$.
In view of the definition (2.6) of a_{KL}^{n+1}, and of the fact that $a(u) = 0$ if $u \le 0$, one has $a_{KL}^{n+1} = 0$, if $\Lambda_{KL} \le 0$. Therefore, the second term in the left hand side of equation (3.9) reads to

$$-\sum_{\sigma_{KL}\in\mathcal{E}_K} a_{KL}^{n+1}\left(\Lambda_{KL}\right)^+\left(u_K^{n+1}-u_L^{n+1}\right)\left(u_K^{n+1}\right)^- \ge 0,$$

Let us now focus on the third term of equation (3.9), and denote by $\mathcal{A}$ this term. Since $a_{KL}^{n+1} = 0$ for $\Lambda_{KL} \le 0$, then $\mathcal{A}$ rewrites

$$\mathcal{A} = \sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}^+\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1}-v_L^{n+1}\right)^+\left(u_K^{n+1}\right)^- - \sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}^+\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1}-v_L^{n+1}\right)^-\left(u_K^{n+1}\right)^-.$$

The second term of $\mathcal{A}$ is nonpositive, but in view of Definition 2.1 on the approximation μ_{KL}^{n+1} and in view of the extension by zero of the function μ for $u \le 0$ since $\mu(0) = 0$, one can deduce that

$$\mu_{KL}^{n+1}\Lambda_{KL}^+\left(v_K^{n+1}-v_L^{n+1}\right)^-\left(u_K^{n+1}\right)^- \le \mu\left(u_K^{n+1}\right)\Lambda_{KL}^+\left(v_K^{n+1}-v_L^{n+1}\right)^-\left(u_K^{n+1}\right)^- = 0,$$

thus, the second term of $\mathcal{A}$ is equal to zero and consequently $\mathcal{A} \ge 0$ since the first term of $\mathcal{A}$ is nonnegative.
Finally, we use the identity $u_K^{n+1} = \left(u_K^{n+1}\right)^+ - \left(u_K^{n+1}\right)^-$ and the nonnegativity of u_K^n, one can deduce from equation (3.9) that $\left(u_K^{n+1}\right)^- = 0$. According to the choice of the dual control volume ω_K, then $\min_{L\in\mathcal{V}}\{u_L^{n+1}\}$ is non-negative. Consequently, $u_K^n \ge 0, \quad \forall K \in \mathcal{V},$ and all $n \in \{0,\ldots,N+1\}$.
In order to prove by induction that $u_K^n \le 1$, $\forall K \in \mathcal{V}$, $\forall n \in \{0,\ldots,N+1\}$, we proceed in the same way as before, so that we consider a dual control volume ω_K such that $u_K^{n+1} = \max_{L\in\mathcal{V}}\{u_L^{n+1}\}$. We get up the result using Remark 2.1, the extension by zero of each of the function a, and μ for $u \ge 1$, and the extension by $f(1) \le 0$ of the continuous function f for $u \ge 1$. ■

3.2. Entropy estimates on $v_{\mathcal{M},\Delta t}$

In the following, C denotes a "generic" constant, that may vary throughout the proofs. We prove now an entropy estimate on $v_{\mathcal{M},\Delta t}$.

Lemma 3.5. *There exists $C > 0$ depending only on $\|v_0\|_{L^2(\Omega)}$, Ω, t_{f}, α and β such that, for all $n^\star \in \{0, \dots, N\}$, one has*

$$\sum_{K\in\mathcal{V}} m_K \Gamma(v_K^{n^\star+1}) + \sum_{n=0}^{n^\star} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2 \le C.$$

Proof. It follows from Jensen's inequality – recall that Γ is convex – that

$$\sum_{K\in\mathcal{V}} m_K \Gamma\left(v_K^0\right) \le \int_\Omega \Gamma\left(v_0(\mathbf{x})\right) \mathrm{d}\mathbf{x}.$$

Since $\Gamma(v) \le (v-1)^2$ for all $v \ge 0$, we obtain that

$$\sum_{K\in\mathcal{V}} m_K \Gamma\left(v_K^0\right) \le \int_\Omega (v_0(\mathbf{x}) - 1)^2 \, \mathrm{d}\mathbf{x} \le C. \tag{3.10}$$

Multiplying the scheme (2.14) by $p(v_K^{n+1})\Delta t$ and summing of $K \in \mathcal{V}$ and $n = 0, \dots, n^\star$ provides

$$\mathcal{A} + \mathcal{B} = \mathcal{C}, \tag{3.11}$$

where we have set

$$\mathcal{A} = \sum_{n=0}^{n^\star} \sum_{K\in\mathcal{V}} m_K (v_K^{n+1} - v_K^n) p(v_K^{n+1}),$$

$$\mathcal{B} = \sum_{n=0}^{n^\star} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2,$$

$$\mathcal{C} = \sum_{n=0}^{n^\star} \Delta t \sum_{K\in\mathcal{V}} m_K (\alpha u_K^n - \beta v_K^{n+1}) p(v_K^{n+1}).$$

Since, thanks to Lemma 3.4, u_K^n is non-negative for all $K \in \mathcal{V}$ and all $n \ge 0$, and since $p(v) \le (v-1)$ for all $v \ge 0$ (with the convention $p(0) = -\infty$), one has

$$\alpha u_K^n p(v_K^{n+1}) \le \alpha u_K^n (v_K^{n+1} - 1).$$

On the other hand, there exists an absolute constant $c^\star$ such that $vp(v) \ge (v-1)^2 - c^\star$ for all $v \ge 0$. Therefore,

$$\beta v_K^{n+1} p(v_K^{n+1}) \ge \beta (v_K^{n+1} - 1)^2 - c^\star.$$

As a consequence, we obtain that

$$\mathcal{C} \le t_{\mathrm{f}} |\Omega| c^\star + \sum_{n=0}^{n^\star} \Delta t \sum_{K\in\mathcal{V}} m_K \left(\alpha u_K^n (v_K^{n+1} - 1) - \beta (v_K^{n+1} - 1)^2\right).$$

Using the weighted Young's inequality $\alpha ab \le \beta b^2 + \frac{\alpha^2}{4\beta} a^2$ for all $(a, b) \in \mathbb{R}^2$ provides

$$\alpha u_K^n (v_K^{n+1} - 1) - \beta (v_K^{n+1} - 1)^2 \le \frac{\alpha^2}{4\beta} u_K^n \le \frac{\alpha^2}{4\beta}$$

thanks to Lemma 3.4. Hence, we obtain that

$$\mathcal{C} \le t_{\mathrm{f}} |\Omega| \left(c^\star + \frac{\alpha^2}{4\beta}\right). \tag{3.12}$$

The function p being increasing, an elementary convexity inequality provides that

$$(a - b)p(a) \ge \Gamma(a) - \Gamma(b), \quad \forall (a, b) \in (\mathbb{R}_+)^2,$$

ensuring that

$$\mathcal{A} \geq \sum_{n=0}^{n^\star} \sum_{K\in\mathcal{V}} m_K \left(\Gamma(v_K^{n+1}) - \Gamma(v_K^n)\right) = \sum_{K\in\mathcal{V}} m_K \left(\Gamma(v_K^{n^\star+1}) - \Gamma(v_K^0)\right). \tag{3.13}$$

Using (3.12), (3.13) and (3.10) in (3.11) concludes the proof of Lemma 3.5. ∎

As a second step, we propose to derive a classical energy estimate on $v_{\mathcal{M},\Delta t}$. Even though the convergence analysis of the scheme can be performed without this estimate, it is interesting to check that our nonlinear scheme also allows to recover the classical estimates ones expects from the classical scheme (2.8).

Lemma 3.6. *There exists C depending only on Ω, $\|v_0\|_{L^2(\Omega)}$, α, β, and t_{f} such that, for all $n^\star \in \{0,\dots,N\}$, one has*

$$\frac{1}{2}\sum_{K\in\mathcal{V}} m_K \left(v_K^{n^\star+1}\right)^2 + \sum_{n=0}^{n^\star} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\left(v_K^{n+1} - v_L^{n+1}\right)^2 \leq C.$$

Proof. Let $n \in \{0,\dots,n^\star\}$, then multiplying the scheme (2.14) by $v_K^{n+1}\Delta t$ and summing over $K \in \mathcal{V}$ yields

$$A^{n+1} + B^{n+1} = C^{n+1} \tag{3.14}$$

where

$$\begin{aligned}
A^{n+1} &= \sum_{K\in\mathcal{V}} m_K v_K^{n+1}\left(v_K^{n+1} - v_K^n\right),\\
B^{n+1} &= \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\eta_{KL}^{n+1}\left(p(v_K^{n+1}) - p(v_L^{n+1})\right)\left(v_K^{n+1} - v_L^{n+1}\right),\\
C^{n+1} &= \Delta t \sum_{K\in\mathcal{V}} m_K \left(\alpha u_K^{n+1} - \beta v_K^{n+1}\right) v_K^{n+1}.
\end{aligned}$$

It follows from the simple inequality $a(a-b) \geq \frac{a^2}{2} - \frac{b^2}{2}$ that

$$A^{n+1} \geq \frac{1}{2}\sum_{K\in\mathcal{V}} m_K \left(v_K^{n+1}\right)^2 - \frac{1}{2}\sum_{K\in\mathcal{V}} m_K \left(v_K^{n+1}\right)^2. \tag{3.15}$$

The definition (2.15) of η_{KL}^{n+1} and the relation (2.10) between η and p implies that

$$D_{KL}\eta_{KL}^{n+1}\left(p(v_K^{n+1}) - p(v_L^{n+1})\right)\left(v_K^{n+1} - v_L^{n+1}\right) \geq D_{KL}\left(v_K^{n+1} - v_L^{n+1}\right)^2, \qquad \forall \sigma_{KL} \in \mathcal{E}.$$

Therefore,

$$B^{n+1} \geq \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\left(v_K^{n+1} - v_L^{n+1}\right)^2. \tag{3.16}$$

Let us now focus on the term C^{n+1}. Thanks to the simple inequality $\alpha ab \leq \frac{\alpha^2}{4\beta}a^2 + \beta b^2$, one gets that

$$C^{n+1} \leq \Delta t \sum_{K\in\mathcal{V}} m_K \frac{\alpha^2}{4\beta}\left(u_K^{n+1}\right)^2.$$

Using now the fact that $0 \leq u_K^{n+1} \leq 1$ (cf. Lemma 3.4), we obtain that

$$C^{n+1} \leq \frac{\alpha^2 \Delta t |\Omega|}{4\beta}. \tag{3.17}$$

Combining (3.15)–(3.17) in (3.14) and summing over $n \in \{0, \dots, N\}$ provides that

$$\frac{1}{2}\sum_{K\in\mathcal{V}} m_K \left(v_K^{n^\star+1}\right)^2 + \sum_{n=0}^{n^\star} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\left(v_K^{n+1} - v_L^{n+1}\right)^2 \leq \frac{1}{2}\sum_{K\in\mathcal{V}} m_K\left(v_K^0\right)^2 + \frac{\alpha^2 n^\star \Delta t|\Omega|}{4\beta}.$$

In order to conclude the proof of Lemma 3.6, it only remains to check that

$$\sum_{K\in\mathcal{V}} m_K\left(v_K^0\right)^2 \leq \|v_0\|^2_{L^2(\Omega)}$$

as a consequence of Jensen's inequality. ■

3.3. **Energy estimates on $u_{\mathcal{M},\Delta t}$**

Proposition 3.7. *There exists a constant $C > 0$ depending only on $\|v_0\|_{L^2(\Omega)}$, Ω, t_{f}, α, β, Λ, D, and $\theta_{\mathcal{T}}$ such that, for all $n^\star \in \{0, \dots, N\}$, one has*

$$\sum_{K\in\mathcal{V}} m_K\left(u_K^{n^\star+1}\right)^2 + \sum_{n=0}^{n^\star}\Delta t \sum_{\sigma_{KL}\in\mathcal{E}} \Lambda_{KL} a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right)^2 \leq C. \tag{3.18}$$

Proof. We multiply equation (2.7) by $\Delta t\, u_K^{n+1}$ and sum over $K \in \mathcal{V}$ and $n \in \{0, \dots, n^\star\}$. This yields

$$E_1 + E_2 + E_3 = E_4, \tag{3.19}$$

where

$$E_1 = \sum_{n=0}^{n^\star}\sum_{K\in\mathcal{V}} m_K\left(u_K^{n+1} - u_K^n\right)u_K^{n+1}, \quad E_2 = \sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K} \Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right)u_K^{n+1},$$

$$E_4 = \sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}} m_K f\left(u_K^{n+1}\right)u_K^{n+1}, \quad E_3 = -\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1} - v_L^{n+1}\right)u_K^{n+1}.$$

For the time evolution term, we use the following inequality: $(a-b)\,a \geq \frac{1}{2}\left(a^2 - b^2\right)$, $\forall a, b \in \mathbb{R}$, to get

$$E_1 \geq \frac{1}{2}\sum_{n=0}^{n^\star}\sum_{K\in\mathcal{V}} m_K\left(\left(u_K^{n+1}\right)^2 - (u_K^n)^2\right) = \frac{1}{2}\sum_{K\in\mathcal{V}} m_K\left(\left(u_K^{n^\star+1}\right)^2 - \left(u_K^0\right)^2\right). \tag{3.20}$$

Next, for the diffusion term, we reorganize the sum over the edges, we find

$$E_2 = \sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right)u_K^{n+1} = \sum_{n=0}^{n^\star}\Delta t\sum_{\sigma_{KL}\in\mathcal{E}}\Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right)^2. \tag{3.21}$$

Similarly, we reorganize the sum over the edges for the convection term. We obtain

$$\begin{aligned} E_3 &= -\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1} - v_L^{n+1}\right)u_K^{n+1} \\ &= -\sum_{n=0}^{n^\star}\Delta t\sum_{\sigma_{KL}\in\mathcal{E}}\Lambda_{KL}\mu_{KL}^{n+1}a_{KL}^{n+1}\left(v_K^{n+1} - v_L^{n+1}\right)\left(u_K^{n+1} - u_L^{n+1}\right). \end{aligned}$$

Using the weighted Young inequality and the uniform boundedness of the function μ, we deduce

$$\begin{aligned}|E_3| &\leq C\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}|\Lambda_{KL}|\,a_{KL}^{n+1}\left|v_K^{n+1}-v_L^{n+1}\right|\left|u_K^{n+1}-u_L^{n+1}\right| \\ &\leq C\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}|\Lambda_{KL}|\left(v_K^{n+1}-v_L^{n+1}\right)^2 \\ &\qquad + \frac{1}{2C_1}\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}|\Lambda_{KL}|\,a_{KL}^{n+1}\left(u_K^{n+1}-u_L^{n+1}\right)^2,\end{aligned}$$

where C_1 is the same constant introduced in Lemma 3.3.
Thanks to estimates (3.3) and (3.5), one has

$$|E_3| \leq C\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}\left(v_K^{n+1}-v_L^{n+1}\right)^2 + \frac{1}{2}\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1}-u_L^{n+1}\right)^2.$$

Therefore, Lemma 3.6 provides

$$|E_3| \leq C + \frac{1}{2}\sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}}\sum_{\sigma_{KL}\in\mathcal{E}_K}\Lambda_{KL}a_{KL}^{n+1}\left(u_K^{n+1}-u_L^{n+1}\right)^2. \tag{3.22}$$

Finally, for the reaction term, since $0\leq u_K^{n+1}\leq 1$ thanks to Lemma 3.4, one has

$$E_4 = \sum_{n=0}^{n^\star}\Delta t\sum_{K\in\mathcal{V}_m} m_K f\left(u_K^{n+1}\right)u_K^{n+1} \leq |\Omega|\,\|f\|_{L^\infty(0,1)}\,t_{\mathrm{f}}. \tag{3.23}$$

Plugging estimates (3.20)–(3.23) into equation (3.19), one deduces that estimate (3.18) holds. ■

3.4. **Enhanced estimate on** $v_{\mathcal{M},\Delta t}$

The goal of this section is to prove a refined estimate on $v_{\mathcal{M},\Delta t}$ inspired from [9, Lemma 3.10], claiming that either $v_{\mathcal{M},\Delta t}$ is constant equal to 0, or $v_{\mathcal{M},\Delta t}\geq r_h>0$ for some r_h depending on the discretization parameters. The first step consists of bounding from below the $L^\infty((0,t_{\mathrm{f}});L^1(\Omega))$ norm of $v_{\mathcal{M},\Delta t}$.

Lemma 3.8. *Assume that $\int_\Omega u_0(\mathbf{x})\mathrm{d}\mathbf{x}>0$ or $\int_\Omega v_0(\mathbf{x})\mathrm{d}\mathbf{x}>0$, then there exists $\kappa>0$ depending on the discretization and on the data such that*

$$\int_\Omega v_{\mathcal{M},\Delta t}(\mathbf{x},t)\mathrm{d}\mathbf{x}\geq\kappa,\quad \forall t\in[0,t_{\mathrm{f}}].$$

Proof. Summing equation (2.14) over $K\in V$ ensures that

$$\sum_{K\in\mathcal{V}} m_K(1+\beta\Delta t)v_K^{n+1} = \sum_{K\in\mathcal{V}} m_K v_K^n + \alpha\Delta t\sum_{K\in\mathcal{V}} m_K u_K^n,\quad \forall n\in\{0,\dots,N\}. \tag{3.24}$$

Assume that $v_{K_\star^n}^n>0$ or $u_{K_\star^n}^n>0$ for some $K_\star^n\in\mathcal{V}$, as this is the case for $n=0$ because of the assumption on the initial data u_0 and v_0, then we deduce from (3.24) and from the non-negativity of v_K^n and u_K^n proved in Lemma 3.4 that

$$\sum_{K\in\mathcal{V}} m_K(1+\beta\Delta t)v_K^{n+1}>0.$$

In particular, there exists $K_\star^{n+1}\in\mathcal{V}$ such that $v_{K_\star^{n+1}}^{n+1}$ is (strictly) positive and

$$\sum_{K\in\mathcal{V}} m_K v_K^{n+1} := \kappa_{n+1}>0.$$

One concludes the proof by setting $\kappa=\min\limits_{n=1,\dots,N+1}\kappa_n$. ■

We give now the definition of D-transmissive path, which was introduced in [9, Definition 3.4].

Definition 3.9. A *D-transmissive path* w joining $K_{\rm i} \in \mathcal{V}$ to $K_{\rm f} \in \mathcal{V}$ consists in a list of vertices $(K_q)_{0\le q\le M}$ such that $K_{\rm i} = K_0$, $K_{\rm f} = K_M$, with $K_q \neq K_\ell$ if $q \neq \ell$, and such that $\sigma_{K_qK_{q+1}} \in \mathcal{E}$ with $D_{K_qK_{q+1}} > 0$ for all $q \in \{0, \dots, M-1\}$. We denote by $\mathcal{W}(K_{\rm i}, K_{\rm f})$ the set of the transmissive path joining $K_{\rm i} \in \mathcal{V}$ to $K_{\rm f} \in \mathcal{V}$.

We now state a result which is proved in [9, Lemma 3.5].

Lemma 3.10. *For all $(K_{\rm i}, K_{\rm f}) \in \mathcal{V}^2$ there exists a transmissive path* w $\in \mathcal{W}(K_{\rm i}, K_{\rm f})$.

We have now introduced all the necessary tools for proving the main result of this section.

Lemma 3.11. *Assume that $\int_\Omega u_0(\mathbf{x})\mathrm{d}\mathbf{x} > 0$ or $\int_\Omega v_0(\mathbf{x})\mathrm{d}\mathbf{x} > 0$, then there exists $r_h > 0$ depending on the data as well as on the mesh $\mathcal{T}$ and Δt such that*

$$v_K^{n+1} \ge r_h, \quad \forall K \in \mathcal{V}, \forall n \in \{0, \dots, N\}. \tag{3.25}$$

Proof. Thanks to Lemma 3.8, we know that there exists $K_{\rm i}$ such that $v_{K_{\rm i}}^{n+1} > 0$. Let $K_{\rm f} \in \mathcal{V}$, then there exists a D-transmissive path w $= (K_q)_{0\le q\le M} \in \mathcal{W}(K_{\rm i}, K_{\rm f})$ thanks to Lemma 3.10, with $K_0 = K_{\rm i}$ and $K_M = K_{\rm f}$.
Thanks to Lemmas 3.3 and 3.5, we know that there exists C such that

$$\sum_{n=0}^{N} \Delta t \sum_{\sigma_{KL}\in\mathcal{E}} |D_{KL}|\, \eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1})\right)^2 \le C.$$

In particular, this ensures that

$$D_{K_qK_{q+1}} \eta_{K_qK_{q+1}}^{n+1} \left(p(v_{K_q}^{n+1}) - p(v_{q+1}^{n+1})\right)^2 \le \frac{C}{\Delta t}, \quad \forall q \in \{0, \dots, M-1\}.$$

Assume now that $v_{K_q}^{n+1} > 0$, as this is the case for $q = 0$, then $\eta_{K_qK_{q+1}}^{n+1} \ge \eta(v_{K_q}^{n+1}) > 0$. Then one has

$$\left(p(v_{K_q}^{n+1}) - p(v_{K_{q+1}}^{n+1})\right)^2 \le \frac{C}{\Delta t D_{K_qK_{q+1}} \eta_{K_qK_{q+1}}^{n+1}} < \infty. \tag{3.26}$$

We deduce from (3.26) that $p(v_{K_{q+1}}^{n+1}) > -\infty$ and, since $\lim_{v\to 0} p(v) = -\infty$ hence $v_{K_{q+1}}^{n+1} > 0$. A straightforward induction provides that $v_{K_{\rm f}}^{n+1} > 0$, and since $K_{\rm f}$ was chosen arbitrarily, we obtain that

$$v_K^{n+1} > 0, \quad \forall K \in \mathcal{V}.$$

Since the set $\mathcal{V} \times \{0, \dots, N\}$ is finite, we can conclude that there exists r_h such that (3.25) holds. ∎

3.5. Existence of a discrete solution

Proposition 3.12. *Given $(u_K^n, v_K^n)_{K\in\mathcal{V}}$ such that $u_{\mathcal{M},\Delta t}(\cdot, n\Delta t)$ and $v_{\mathcal{M},\Delta t}(\cdot, n\Delta t)$ are non-negative, then there exists (at least) one solution $\left(u_K^{n+1}, v_K^{n+1}\right)_{K\in\mathcal{V}}$ of the scheme* (2.7),(2.14)*. Moreover, $u_{\mathcal{M},\Delta t}(\cdot, n\Delta t)$ and $v_{\mathcal{M},\Delta t}(\cdot, n\Delta t)$ are non-negative.*

Proof. The case where $(u_K^n, v_K^n)_{K\in\mathcal{V}} \equiv 0$ has to be treated apart. In this very particular case, it is easy to check that $\left(u_K^{n+1}, v_K^{n+1}\right)_{K\in\mathcal{V}} \equiv 0$ is a solution to the scheme.
Let us now focus on the case where u_K^n or v_K^n is strictly positive for some $K \in \mathcal{V}$. Because of the weak coupling on the numerical scheme, we can first solve (2.14), and afterwards (2.7). The existence of a solution $\left(v_K^{n+1}\right)_{K\in\mathcal{V}}$ can be proved by slightly adapting the proof of [9, Proposition 3.11], which

relies on a topological argument. The main difficulty comes from the fact the scheme (2.14) is not continuous w.r.t. $\left(v_K^{n+1}\right)_{K\in\mathcal{V}}$ on $(\mathbb{R}_+)^{\#\mathcal{V}}$, but Lemma 3.11 ensures that no component v_K^{n+1} of the discrete solution can go close to 0. Let us detail now the proof.
Let $\gamma \in [0,1]$, we denote by $\left(v_{K,\gamma}^{n+1}\right)_{K\in\mathcal{V}}$ the solution (if it exists) to the numerical scheme

$$\frac{v_{K,\gamma}^{n+1}-v_K^n}{\Delta t}m_K + \gamma \sum_{\sigma_{KL}\in\mathcal{E}_K} D_{KL}\eta_{KL,\gamma}^{n+1}\left(p(v_{K,\gamma}^{n+1})-p(v_{L,\gamma}^{n+1})\right)$$
$$(1-\gamma)\sum_{\sigma_{KL}\in\mathcal{E}_K}|D_{KL}|(p(v_{K,\gamma}^{n+1})-p(v_{L,\gamma}^{n+1})) = \alpha u_K^n m_K - \beta v_{K,\gamma}^{n+1}m_K. \quad (3.27)$$

In the above scheme, we have set

$$\eta_{KL,\gamma}^{n+1} = \begin{cases} \max_{v\in J_{KL,\gamma}^{n+1}} \eta(v) & \text{if } D_{KL}\geq 0,\\ \min_{v\in J_{KL,\gamma}^{n+1}} \eta(v) & \text{if } D_{KL}< 0,\end{cases}$$

where $J_{KL,\gamma}^{n+1} = \left[\min(v_{K,\gamma}^{n+1}, v_{L,\gamma}^{n+1}), \max(v_{K,\gamma}^{n+1}, v_{L,\gamma}^{n+1})\right]$. Reproducing the analysis carried out in §3.2 and §3.4, we get that for all $\gamma\in[0,1]$,

$$\sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\left(\phi(v_{K,\gamma}^{n+1})-\phi(v_{L,\gamma}^{n+1})\right)^2 \leq \sum_{\sigma_{KL}\in\mathcal{E}} D_{KL}\eta_{KL,\gamma}^{n+1}\left(p(v_{K,\gamma}^{n+1})-p(v_{L,\gamma}^{n+1})\right)^2 \leq C \quad (3.28)$$

and, that there exists $\epsilon>0$ such that

$$v_{K,\gamma}^{n+1} \geq \epsilon > 0, \qquad \forall K\in\mathcal{V}. \quad (3.29)$$

This ensures in particular that for all $\gamma\in[0,1]$, the solutions of equation (3.27) stay in the interior of a compact subset $\mathcal{K}$ of $\mathbb{R}^{\#\mathcal{V}}$ such that

$$\operatorname{dist}\left(\mathcal{K}, (\mathbb{R}_-)^{\#\mathcal{V}}\right) \geq \frac{\epsilon}{2}.$$

Define the function $\Upsilon : \mathcal{K}\times[0,1]\to\mathbb{R}^{\#\mathcal{V}}$ by: $\forall K\in\mathcal{V}$,

$$\Upsilon_K\left((w_K)_K,\gamma\right) = \frac{w_K - v_K^n}{\Delta t}m_K + \gamma\sum_{\sigma_{KL}\in\mathcal{E}_K} D_{KL}\eta_{KL,\gamma}^{n+1}\left(p(w_K)-p(w_L)\right)$$
$$+(1-\gamma)\sum_{\sigma_{KL}\in\mathcal{E}_K}|D_{KL}|(w_K)-p(w_L)) - \alpha u_K^n m_K + \beta w_K m_K.$$

The function Υ is uniformly continuous on $\mathcal{K}\times[0,1]$, and for all $\gamma\in[0,1]$ the solution $v_{K,\gamma}^{n+1}$ of the nonlinear system

$$\Upsilon\left(\left(v_{K,\gamma}^{n+1}\right)_{K\in\mathcal{V}},\gamma\right) = 0 \quad (3.30)$$

cannot reach $\partial\mathcal{K}$. For $\gamma=0$, the system is monotone, so that the system (3.30) admits a unique solution, whose topological degree is equal to 1 (we refer to [19, Proposition 3.1] for a proof of this property). The topological degree being constant w.r.t. $\gamma\in[0,1]$, the system (3.30) admits at least one solution for $\gamma=1$, concluding the proof of the existence of $\left(v_K^{n+1}\right)_{K\in\mathcal{V}}$.
The existence proof for $\left(u_K^{n+1}\right)_{K\in\mathcal{V}}$ is similar but simpler since

- **i.** the *a priori estimate* $0\leq u_K^{n+1}\leq 1$ is sufficient for the claim, and no energy estimate is needed here;
- **ii.** the scheme (2.7) depends in a uniformly continuous way on $\left(u_K^{n+1}\right)_{K\in\mathcal{V}}$ on the compact subset $[-1,2]^{\#\mathcal{V}}$ of $\mathcal{R}^{\#\mathcal{V}}$.

Therefore, we let to the reader the care of checking the proof for self-conviction. ■

4. **Compactness estimates on the family of discrete solutions.**

As a consequence of Lemmas 3.4 and 3.6, the sequences $\left(u_{\mathcal{M}_m,\Delta t_m}\right)_m$ and $\left(v_{\mathcal{M}_m,\Delta t_m}\right)_m$ are uniformly bounded w.r.t. m in $L^\infty(Q_{t_\mathrm{f}})$ and $L^\infty(0,t_\mathrm{f};L^2(\Omega))$ respectively. Moreover, as a consequence of Lemma 3.6, the sequence $\left(v_{\mathcal{T}_m,\Delta t_m}\right)_m$ is uniformly bounded in $L^2(0,t_\mathrm{f};H^1(\Omega))$. Therefore, there exists $v \in L^2(0,t_\mathrm{f};H^1(\Omega))$ such that, up to an unlabeled subsequence,

$$v_{\mathcal{T}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} v \quad \text{weakly in } L^2(0,t_\mathrm{f};H^1(\Omega)).$$

we deduce from the following inequality (see for instance [6, Lemma 3.4] or [10, Lemma 6.5])

$$\|w_{\mathcal{T}_m,\Delta t_m} - w_{\mathcal{M}_m,\Delta t_m}\|_{L^2(\Omega)} \le Ch\|\nabla w_{\mathcal{T}_m,\Delta t_m}\|_{L^2(\Omega)}, \qquad \forall w_{\mathcal{T}_m,\Delta t_m} \in \mathcal{H}_{\mathcal{T}_m,\Delta t_m}, \tag{4.1}$$

that $\left(v_{\mathcal{T}_m,\Delta t_m}\right)_m$ and $\left(v_{\mathcal{M}_m,\Delta t_m}\right)_m$ have the same limit, so that, up to an unlabeled subsequence,

$$v_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} v \quad \text{in the } L^\infty(0,t_\mathrm{f};L^2(\Omega))\text{-weak-}\star \text{ sense.}$$

On the other hand, the combination of Lemma 3.1 with Proposition 3.7 provides that

$$\iint_{Q_{t_\mathrm{f}}} \left|\nabla \xi_{\mathcal{T}_m,\Delta t_m}\right|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C$$

for some C not depending on m, where $\xi_{\mathcal{T}_m,\Delta t_m}$ is the piecewise linear function with nodal values $\xi_K^n = \xi(u_K^n)$ for all $K \in \mathcal{V}_m$ and all $n \in \{0,\dots,N_m+1\}$. Therefore, there exists $\xi^\star \in L^2(0,t_\mathrm{f};H^1(\Omega)) \cap L^\infty(Q_{t_\mathrm{f}})$ such that, up to an unlabeled subsequence,

$$\xi_{\mathcal{T}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} \xi^\star \quad \text{weakly in } L^2(0,t_\mathrm{f};H^1(\Omega)).$$

It follows from inequality (4.1) that $\left(\xi_{\mathcal{T}_m,\Delta t_m}\right)_m$ and $\left(\xi_{\mathcal{M}_m,\Delta t_m}\right)_m$ share the same limit, therefore, up to an unlabeled subsequence,

$$\xi_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} \xi^\star \quad \text{in the } L^\infty(Q_{t_\mathrm{f}})\text{-weak-}\star \text{ sense.}$$

Finally, since $\left(u_{\mathcal{M}_m,\Delta t_m}\right)_m$ is uniformly bounded in $L^\infty(Q_{t_\mathrm{f}})$, then there exists $u \in L^\infty(Q_{t_\mathrm{f}})$ such that, up to an unlabeled subsequence,

$$u_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} u \quad \text{in the } L^\infty(Q_{t_\mathrm{f}})\text{-weak-}\star \text{ sense.}$$

The goal of this section is to show that $\xi^\star = \xi(u)$, and that

$$v_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} v \quad \text{a.e. in } Q_{t_\mathrm{f}} \qquad \text{and} \qquad u_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} u \quad \text{a.e. in } Q_{t_\mathrm{f}},$$

As an alternative to the lengthy and technical proof that consists in estimating the time- and space-translates of the discrete functions (see [3] for the continuous framework and [21] for the discrete setting), we make use of the technical blackbox proposed in [5, Theorem 3.9]. We refer to [16, Lemmas 4.4 and 6.6], [17, §4.2, §C.1.6], and [23] for alternative but very close approaches.

Let $m \ge 1$ be fixed, then let us denote by $\left(\varphi_K^{n+1}\right)_{K\in\mathcal{V}_m, 0\le n\le N_m}$ a set a nodal values such that $\varphi_K^{n+1} = 0$ if $\mathbf{x}_K \in \partial\Omega$. We deduce the functions $\varphi_{\mathcal{T}_m,\Delta t_m}$ and $\varphi_{\mathcal{M}_m,\Delta t_m}$. We state now discrete $L^1\left(0,t_\mathrm{f};\left(H^1(\Omega)\right)'\right)$ estimates on the finite differences w.r.t. time of $u_{\mathcal{M}_m,\Delta t_m}$ and $v_{\mathcal{M}_m,\Delta t_m}$.

Lemma 4.1. *There exists C not depending on m such that*

$$\sum_{n=0}^{N_m} \sum_{K \in \mathcal{V}_m} m_K \left(u_K^{n+1} - u_K^n \right) \varphi_K^{n+1} \leq C \| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \|_{L^2(Q_{t_f})}, \tag{4.2}$$

$$\sum_{n=0}^{N_m} \sum_{K \in \mathcal{V}_m} m_K \left(v_K^{n+1} - v_K^n \right) \varphi_K^{n+1} \leq C \| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \|_{L^2(Q_{t_f})}. \tag{4.3}$$

Proof. We only establish (4.3) since the proof of (4.2) is similar. Multiplying (2.14) by $\Delta t \varphi_K^{n+1}$ and summing over $n \in \{0, \dots, N_m\}$ and $K \in \mathcal{V}_m$ yields

$$\sum_{n=0}^{N_m} \sum_{K \in \mathcal{V}_m} m_K \left(v_K^{n+1} - v_K^n \right) \varphi_K^{n+1} \leq A_m + B_m, \tag{4.4}$$

where

$$A_m = - \sum_{n=0}^{N_m} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} D_{KL} \eta_{KL}^{n+1} \left(p(v_K^{n+1}) - p(v_L^{n+1}) \right) \left(\varphi_K^{n+1} - \varphi_L^{n+1} \right),$$

$$B_m = \sum_{n=0}^{N_m} \Delta t \sum_{K \in \mathcal{V}_m} m_K \left(\alpha u_K^{n+1} - \beta v_K^{n+1} \right) \varphi_K^{n+1}.$$

It follows from Cauchy-Schwarz inequality and from $\|\eta\|_\infty = 1$ that

$$|A_m|^2 \leq \left(\sum_{n=0}^{N_m} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} |D_{KL}| \eta_{KL} \left(p(v_K^{n+1}) - p(v_L^{n+1}) \right)^2 \right) \left(\sum_{n=0}^{N_m} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} |D_{KL}| \left(\varphi_K^{n+1} - \varphi_L^{n+1} \right)^2 \right)$$

Combining Lemmas 3.3 and 3.5 provides

$$\sum_{n=0}^{N_m} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} |D_{KL}| \eta_{KL} \left(p(v_K^{n+1}) - p(v_L^{n+1}) \right)^2 \leq C,$$

whereas Lemma 3.2 implies that

$$\sum_{n=0}^{N_m} \Delta t \sum_{\sigma_{KL} \in \mathcal{E}} |D_{KL}| \left(\varphi_K^{n+1} - \varphi_L^{n+1} \right)^2 \leq C \left\| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})}^2.$$

Therefore, we obtain that

$$|A_m| \leq C \left\| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})} \tag{4.5}$$

On the other hand, Cauchy-Schwarz inequality provides

$$|B_m| \leq \left\| \alpha u_{\mathcal{M}_m, \Delta t_m} - \beta v_{\mathcal{M}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})} \left\| \varphi_{\mathcal{M}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})}.$$

It results from Proposition 3.7 and Lemma 3.6 that

$$\left\| \alpha u_{\mathcal{M}_m, \Delta t_m} - \beta v_{\mathcal{M}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})} \leq C,$$

whereas the discrete Poincaré's inequality [6, Lemma 3.3] ensures that

$$\left\| \varphi_{\mathcal{M}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})} \leq C \left\| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})}.$$

Gathering the previous inequalities, one gets that

$$|B_m| \leq C \left\| \nabla \varphi_{\mathcal{T}_m, \Delta t_m} \right\|_{L^2(Q_{t_f})}. \tag{4.6}$$

Putting (4.4), (4.5) and (4.6) together provides (4.3). ∎

We have all the necessary estimates at hand to make use of [5, Theorem 3.9]. It allows us to claim directly that $\xi^\star = \xi(u)$, and that

$$v_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} v \quad \text{a.e. in } Q_{t_\mathrm{f}} \qquad \text{and} \qquad u_{\mathcal{M}_m,\Delta t_m} \underset{m\to\infty}{\longrightarrow} u \quad \text{a.e. in } Q_{t_\mathrm{f}}.$$

5. Identification as a weak solution

It remains to be shown that (u, v) satisfies the weak formulation (1.6)–(1.7). To do this, we consider a test function $\psi \in \mathcal{D}\left(\overline{\Omega} \times [0, t_\mathrm{f})\right)$, and denote by $\psi_K^n = \psi(\mathbf{x}_K, t^n)$, for all $K \in \mathcal{V}_m$ and all $n \in \{0, \dots, N_m\}$. Let us focus on the convergence of the first equation of scheme (2.7)–(2.14), i.e., we show that equation (1.6) is verified when $m \to \infty$. We note that the convergence of the second equation of the scheme is similar and major difficulties that we can encounter are discussed hereafter.

Multiplying the first equation (2.7) by $\Delta t_m \psi_K^n$ and summing over $n \in \{0, \dots, N_m\}$ and $K \in \mathcal{V}_m$ yields, after a reorganization of the sum,

$$\mathcal{A}_m + \mathcal{B}_m + \mathcal{C}_m + \mathcal{D}_m = \mathcal{F}_m, \tag{5.1}$$

where

$$\mathcal{A}_m = \sum_{n=0}^{N_m} \sum_{K\in\mathcal{V}_m} \left(u_K^{n+1} - u_K^n\right) \psi_K^n m_K, \qquad \mathcal{F}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{K\in\mathcal{V}_m} f\left(u_K^{n+1}\right) \psi_K^n m_K,$$

$$\mathcal{B}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} \Lambda_{KL} \left(a_{KL}^{n+1}\left(u_K^{n+1} - u_L^{n+1}\right) - \sqrt{a_{KL}^{n+1}}\left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)\right)\left(\psi_K^n - \psi_L^n\right),$$

$$\mathcal{C}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} \Lambda_{KL} \sqrt{a_{KL}^{n+1}}\left(\xi\left(u_K^{n+1}\right) - \xi\left(u_L^{n+1}\right)\right)\left(\psi_K^n - \psi_L^n\right),$$

$$\mathcal{D}_m = -\sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} \Lambda_{KL} \mu_{KL}^{n+1} a_{KL}^{n+1}\left(v_K^{n+1} - v_L^{n+1}\right)\left(\psi_K^n - \psi_L^n\right).$$

Accumulation term

Note that $\psi_K^{N_m+1} = 0$ for all $K \in \mathcal{V}_m$, then, performing summation by parts in time, the term $\mathcal{A}_m$ can be rewritten

$$\begin{aligned}
\mathcal{A}_m &= \sum_{n=0}^{N_m} \sum_{K\in\mathcal{V}_m} u_K^{n+1}\psi_K^n m_K - \sum_{n=1}^{N_m} \sum_{K\in\mathcal{V}_m} u_K^n \psi_K^n m_K - \sum_{K\in\mathcal{V}_m} u_K^0 \psi_K^0 m_K \\
&= -\sum_{n=0}^{N_m} \Delta t_m \sum_{K\in\mathcal{V}_m} u_K^{n+1} \frac{\psi_K^{n+1} - \psi_K^n}{\Delta t_m} m_K - \sum_{K\in\mathcal{V}_m} u_K^0 \psi_K^0 m_K \\
&= -\iint_{Q_{t_\mathrm{f}}} u_{\mathcal{M}_m,\Delta t_m}(\mathbf{x}, t)\, \partial_t \psi_{\mathcal{M}_m,\Delta t_m}(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int_\Omega u_{\mathcal{M}_m,\Delta t_m}(\mathbf{x}, 0)\, \psi_{\mathcal{M}_m,\Delta t_m}(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x}.
\end{aligned}$$

Thanks to the regularity of ψ, and the convergence in $L^1(Q_{t_\mathrm{f}})$ of the sequence $\left(u_{\mathcal{M}_m,\Delta t_m}\right)_m$ towards u, it follows that (see e.g. [20])

$$\mathcal{A}_m \longrightarrow -\iint_{Q_{t_\mathrm{f}}} u(\mathbf{x}, t)\, \partial_t \psi(\mathbf{x}, t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int_\Omega u(\mathbf{x}, 0)\, \psi(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x}, \qquad \text{as } m \to \infty.$$

Diffusion term

Let us first prove that $\lim_{m\to\infty} \mathcal{B}_m = 0$.
For all $\sigma_{KL} \in \mathcal{E}_m$ and all $n \in \{0, \dots, N_m\}$, we denote by $\overline{a}_{KL}^{n+1}$ the quantity defined by

$$\overline{a}_{KL}^{n+1} = \begin{cases} \left(\dfrac{\xi(u_K^{n+1}) - \xi(u_L^{n+1})}{u_K^{n+1} - u_L^{n+1}} \right)^2 & \text{if } u_K^{n+1} \neq u_L^{n+1}, \\ a\left(u_K^{n+1}\right) & \text{if } u_K^{n+1} = u_L^{n+1}. \end{cases}$$

Then, the term $\mathcal{B}_m$ rewrites

$$\mathcal{B}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} \Lambda_{KL} \sqrt{a_{KL}^{n+1}} \left(\sqrt{a_{KL}^{n+1}} - \sqrt{\overline{a}_{KL}^{n+1}} \right) \left(u_K^{n+1} - u_L^{n+1} \right) \left(\psi_K^n - \psi_L^n \right).$$

Now, using the Cauchy-Schwarz inequality, we get

$$|\mathcal{B}_m| \leq \left(\sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} |\Lambda_{KL}|\, a_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1} \right)^2 \right)^{\frac{1}{2}} \times \mathcal{R}_m{}^{\frac{1}{2}},$$

where, $\mathcal{R}_m$ is given by

$$\mathcal{R}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{\sigma_{KL}\in\mathcal{E}_m} |\Lambda_{KL}| \left(\sqrt{a_{KL}^{n+1}} - \sqrt{\overline{a}_{KL}^{n+1}} \right)^2 \left(\psi_K^n - \psi_L^n \right)^2.$$

Using Lemma 3.3 and Proposition 3.7, one has $|\mathcal{B}_m| \leq C \mathcal{R}_m{}^{\frac{1}{2}}$. Hence, in order to prove that $\lim_{m\to\infty} \mathcal{B}_m = 0$, it suffices to prove that $\lim_{m\to\infty} \mathcal{R}_m = 0$.
For all $T \in \mathcal{T}_m$, we denote by

$$\overline{\xi}_T^{n+1} = \max_{\mathbf{x}\in T} \left(\xi\left(p\right)_{\mathcal{T}_m,\Delta t_m} \left(\mathbf{x}, t^{n+1}\right) \right), \qquad \underline{\xi}_T^{n+1} = \min_{\mathbf{x}\in T} \left(\xi\left(p\right)_{\mathcal{T}_m,\Delta t_m} \left(\mathbf{x}, t^{n+1}\right) \right),$$

and for all $(\mathbf{x}, t) \in T \times \left(t^n, t^{n+1}\right)$, by

$$\overline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x}, t) = \overline{\xi}_T^{n+1}, \qquad \underline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x}, t) = \underline{\xi}_T^{n+1}.$$

Consider the uniform continuous function $\sqrt{a \circ \xi^{-1}}$ defined on the closed bounded interval $[0, \xi(1)]$, and let ϱ be its modulus of continuity, then we have

$$\left| \sqrt{a_{KL}^{n+1}} - \sqrt{\overline{a}_{KL}^{n+1}} \right| \leq \varrho\left(\overline{\xi}_T^{n+1} - \underline{\xi}_T^{n+1} \right), \qquad \text{for all } \sigma_{KL} \in \mathcal{E}_T. \tag{5.2}$$

Therefore, using this inequality in the definition of $\mathcal{R}_m$, we get

$$0 \leq \mathcal{R}_m \leq \mathcal{Q}_m \tag{5.3}$$

where,

$$\mathcal{Q}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{T\in\mathcal{T}_m} \left(\varrho\left(\overline{\xi}_T^{n+1} - \underline{\xi}_T^{n+1} \right) \right)^2 \sum_{\sigma_{KL}\in\mathcal{E}_T} \left| \lambda_{KL}^T \right| \left(\psi_K^n - \psi_L^n \right)^2, \tag{5.4}$$

and λ_{KL}^T is the constant defined by (3.3).
Thanks to Lemma 3.2, one can deduce that the inequality (5.3) implies that

$$0 \leq \mathcal{R}_m \leq C \iint_{Q_{t_f}} \varrho\left(\overline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x}, t) - \underline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x}, t) \right) \mathrm{d}\mathbf{x}\, \mathrm{d}t,$$

where C is independent of h_m, and Δt_m. Therefore, it suffices to show that $\overline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) - \underline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) \to 0$ a.e. in $Q_{t_{\rm f}}$ to consequently prove that $\lim\limits_{m\to\infty} \mathcal{R}_m = 0$. By a simple generalization of [9, Lemma A.1] and by the help of Lemma 3.1 and Proposition 3.7, it follows that

$$\iint_{Q_{t_{\rm f}}} \left|\overline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) - \underline{\xi}_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t)\right| \mathrm{d}\mathbf{x}\,\mathrm{d}t \le Ch \left(\iint_{Q_{t_{\rm f}}} \left|\nabla \xi\left(u\right)_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t\right)^{\frac{1}{2}} \le Ch.$$

As a consequence, up to a subsequence, one has

$$\lim_{m\to\infty} \mathcal{B}_m = \lim_{m\to\infty} \mathcal{R}_m = \lim_{m\to\infty} \mathcal{Q}_m = 0.$$

We now focus on the term $\mathcal{C}_m$ and prove that

$$\lim_{m\to\infty} \mathcal{C}_m = \iint_{Q_{t_{\rm f}}} \Lambda(\mathbf{x})\sqrt{a\left(u\right)}\nabla\xi(u)\cdot\nabla\psi \mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

To do this, we introduce the term $\mathcal{C}'_m$ defined by

$$\mathcal{C}'_m := \iint_{Q_{t_{\rm f}}} \Theta_{\mathcal{T}_m,\Delta t_m}\Lambda(\mathbf{x})\nabla\xi\left(u\right)_{\mathcal{T}_m,\Delta t_m}\cdot\nabla\psi_{\mathcal{T}_m,\Delta t_m}\left(\cdot,t-\Delta t_m\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$

where $\Theta_{\mathcal{T}_m,\Delta t_m}$ is a piecewise constant function (on the triangular mesh) function given by

$$\Theta_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) = \sqrt{a\circ\xi^{-1}}\left(\Upsilon_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t)\right), \qquad \forall \mathbf{x}\in T,\ \forall t\in(t^n,t^{n+1}],\ \forall T\in\mathcal{T}_m,$$

where $\Upsilon_{\mathcal{T}_m,\Delta t_m}$ is defined by

$$\Upsilon_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) = \xi\left(u\right)_{\mathcal{T}_m,\Delta t_m}(\mathbf{x}_T,t), \qquad \forall \mathbf{x}\in T,\ \forall t\in(t^n,t^{n+1}],\ \forall T\in\mathcal{T}_m.$$

Using again a slight generalization of [9, Lemma A.1] as well as the boundedness of the continuous function $\sqrt{a\circ\xi^{-1}}$, we obtain

$$\begin{aligned} \Upsilon_{\mathcal{T}_m,\Delta t_m} &\longrightarrow \xi\left(u\right) && \text{in } L^2\left(Q_{t_{\rm f}}\right) \text{ as } m\to\infty, \\ \Theta_{\mathcal{T}_m,\Delta t_m} &\longrightarrow \sqrt{a\left(u\right)} && \text{in } L^2\left(Q_{t_{\rm f}}\right) \text{ as } m\to\infty. \end{aligned} \tag{5.5}$$

It remains to verify that $|\mathcal{C}_m - \mathcal{C}'_m| \longrightarrow 0$, when m tends to infinity.
We denote by

$$a_T^{n+1} = \left(\Theta_{\mathcal{T}_m,\Delta t_m}\left(\mathbf{x}_T,t^{n+1}\right)\right)^2, \qquad \forall T\in\mathcal{T}_m,\ \forall n\in\{0,\ldots,N_m\}.$$

The discretization of the term $\mathcal{C}'_m$ is written as

$$\mathcal{C}'_m = \sum_{n=0}^{N_m}\Delta t_m\sum_{T\in\mathcal{T}_m}\sqrt{a_T^{n+1}}\sum_{\sigma_{KL}\in\mathcal{E}_T}\lambda_{KL}^T\left(\xi\left(u_K^{n+1}\right)-\xi\left(u_L^{n+1}\right)\right)\left(\psi_K^n-\psi_L^n\right).$$

Similar arguments as for obtaining inequality (5.2) yield

$$\left|\sqrt{a_{KL}^{n+1}}-\sqrt{a_T^{n+1}}\right| \le \varrho\Big(\overline{\xi}_T^{n+1}-\underline{\xi}_T^{n+1}\Big), \qquad \text{for all } \sigma_{KL}\in\mathcal{E}_T.$$

Therefore, using the Cauchy-Schwarz inequality, Lemma 3.1, Lemma 3.2, and Proposition 3.7, we deduce that there exists a constant C does not depend on h_m such that

$$\begin{aligned} |\mathcal{C}_m-\mathcal{C}'_m|^2 &\le \left(\sum_{n=0}^{N_m}\Delta t_m\sum_{T\in\mathcal{T}_m}\varrho\Big(\overline{\xi}_T^{n+1}-\underline{\xi}_T^{n+1}\Big)\sum_{\sigma_{KL}\in\mathcal{E}_T}\left|\lambda_{KL}^T\right|\left|\xi\left(u_K^{n+1}\right)-\xi\left(u_L^{n+1}\right)\right|\left|\psi_K^n-\psi_L^n\right|\right)^2 \\ &\le \mathcal{Q}_m\times\sum_{n=0}^{N_m}\Delta t_m\sum_{T\in\mathcal{T}_m}\sum_{\sigma_{KL}\in\mathcal{E}_T}\left|\lambda_{KL}^T\right|\left|\xi\left(u_K^{n+1}\right)-\xi\left(u_L^{n+1}\right)\right|^2 \le C\mathcal{Q}_m \longrightarrow 0 \quad \text{as } m\to\infty. \end{aligned}$$

Convection term

For all $T \in \mathcal{T}_m$, we define the piecewise constant function $\kappa_{\mathcal{T}_m,\Delta t_m}$ by

$$\kappa_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t) = \chi \circ \xi^{-1}\left(\Upsilon_{\mathcal{T}_m,\Delta t_m}(\mathbf{x},t)\right), \qquad \forall \mathbf{x} \in T,\, \forall t \in (t^n, t^{n+1}].$$

Using the same guidelines as for the convergence results (5.5), one has

$$\kappa_{\mathcal{T}_m,\Delta t_m} \longrightarrow \chi(u) \qquad \text{in } L^2(Q_{t_\mathrm{f}}) \text{ as } m \to \infty.$$

We introduce the term

$$\mathcal{D}'_m := -\iint_{Q_{t_\mathrm{f}}} \kappa_{\mathcal{T}_m,\Delta t_m} \Lambda(\mathbf{x}) \nabla v_{\mathcal{T}_m,\Delta t_m} \cdot \nabla \psi_{\mathcal{T}_m,\Delta t_m}(\cdot, t - \Delta t_m)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t.$$

Thanks to the weak convergence in $L^2(Q_{t_\mathrm{f}})$ of the sequence $\nabla v_{\mathcal{T}_m,\Delta t_m}$ towards ∇v, and to the uniform convergence of $\nabla \psi_{\mathcal{T}_m,\Delta t_m}$ towards $\nabla \psi$, we obtain

$$\mathcal{D}'_m \longrightarrow -\iint_{Q_{t_\mathrm{f}}} \chi(u)\, \Lambda(\mathbf{x}) \nabla v \cdot \nabla \psi \mathrm{d}\mathbf{x}\, \mathrm{d}t \qquad \text{as } m \to \infty.$$

Let us prove, using the same guidelines as before, that $|\mathcal{D}_m - \mathcal{D}'_m| \longrightarrow 0$, when m tends to infinity. We denote by

$$\chi_T^{n+1} = \kappa_{\mathcal{T}_m,\Delta t_m}\left(\mathbf{x}_T, t^{n+1}\right), \qquad \forall T \in \mathcal{T}_m, \forall n \in \{0, \dots, N_m\},$$
$$\mu_T^{n+1} = \mu_{\mathcal{T}_m,\Delta t_m}\left(\mathbf{x}_T, t^{n+1}\right), \qquad \forall T \in \mathcal{T}_m, \forall n \in \{0, \dots, N_m\}.$$

Therefore,

$$\mathcal{D}_m - \mathcal{D}'_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{T \in \mathcal{T}_m} \sum_{\sigma_{KL} \in \mathcal{E}_T} \left(a_T^{n+1}\mu_T^{n+1} - a_{KL}^{n+1}\mu_{KL}^{n+1}\right) \lambda_{KL}^T \left(v_K^{n+1} - v_L^{n+1}\right)\left(\psi_K^n - \psi_L^n\right).$$

Thanks to the the triangle inequality and to the existence of a continuity moduli η and δ of the continuous functions $\sqrt{a \circ \xi^{-1}}$ and $\mu \circ \xi^{-1}$ respectively, one has

$$\begin{aligned}\left|a_{KL}^{n+1}\mu_{KL}^{n+1} - a_T^{n+1}\mu_T^{n+1}\right| &\le \mu_{KL}^{n+1}\left|a_{KL}^{n+1} - a_T^{n+1}\right| + a_T^{n+1}\left(\mu_{KL}^{n+1} - \mu_T^{n+1}\right) \\ &\le C\left(\varrho\Big(\overline{\xi}_T^{n+1} - \underline{\xi}_T^{n+1}\Big) + \delta\Big(\overline{\xi}_T^{n+1} - \underline{\xi}_T^{n+1}\Big)\right),\end{aligned}$$

where the constant C does not depend on h_m. Therefore, using the Cauchy-Schwarz inequality, Lemma 3.1, Lemma 3.2, and Proposition 3.7, we deduce that there exists a constant C independent of h_m such that

$$|\mathcal{D}_m - \mathcal{D}'_m|^2 \le C\left(\mathcal{Q}_m + \mathcal{W}_m\right) \times \sum_{n=0}^{N_m} \Delta t_m \sum_{T \in \mathcal{T}_m} \sum_{\sigma_{KL} \in \mathcal{E}_T} \left|\lambda_{KL}^T\right| \left|v_K^{n+1} - v_L^{n+1}\right|^2,$$

where $\mathcal{Q}_m$ is given by equation (5.4), and $\mathcal{W}_m$ is given by

$$\mathcal{W}_m = \sum_{n=0}^{N_m} \Delta t_m \sum_{T \in \mathcal{T}_m} \left(\delta\Big(\overline{\xi}_T^{n+1} - \underline{\xi}_T^{n+1}\Big)\right)^2 \sum_{\sigma_{KL} \in \mathcal{E}_T} \left|\lambda_{KL}^T\right| \left(\psi_K^n - \psi_L^n\right)^2.$$

Now, using the same proof as for the diffusion term, one can deuce that $\mathcal{W}_m \le C h_m$. Therefore

$$\lim_{m \to \infty} |\mathcal{D}_m - \mathcal{D}'_m| = 0,$$

and consequently,

$$\lim_{m \to \infty} \mathcal{D}_m = -\iint_{Q_{t_\mathrm{f}}} \chi(u)\, \Lambda(\mathbf{x}) \nabla v \cdot \nabla \psi \mathrm{d}\mathbf{x}\, \mathrm{d}t.$$

Reaction term

We would now like to show that

$$\mathcal{F}_m \longrightarrow \iint_{Q_{t_\mathrm{f}}} f\left(u\left(\mathbf{x},t\right)\right)\psi\left(\mathbf{x},t\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad \text{as } m\to\infty.$$

For this purpose, we denote, for all $K\in\mathcal{V}_m$ and for all $n\geq 1$, by $f_K^n = f\left(u_K^n\right)$, and by $f_{\mathcal{M}_m,\Delta t_m}$ the piecewise constant reconstruction in $\mathcal{X}_{\mathcal{M}_m,\Delta t_m}$. Thus we have

$$\mathcal{F}_m = \iint_{Q_{t_\mathrm{f}}} f_{\mathcal{M}_m,\Delta t_m}\psi_{\mathcal{M}_m,\Delta t_m}\left(\cdot, t-\Delta t_m\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \longrightarrow \iint_{Q_{t_\mathrm{f}}} f\left(u\left(\mathbf{x},t\right)\right)\psi\left(\mathbf{x},t\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad \text{as } m\to\infty,$$

since $f\left(u\right)_{\mathcal{M}_m,\Delta t_m}$ converges strongly in $L^2\left(Q_{t_\mathrm{f}}\right)$ towards $f\left(u\right)$, and as $\psi_{\mathcal{M}_m,\Delta t_m}$ converges uniformly towards ψ. This ends the proof of Theorem 2.2.

6. Numerical results

In this section, we establish various 2–D numerical results provided by the *nonlinear CVFE scheme* (2.7), (2.14). Newton's algorithm is carried out for the implementation of the scheme, coupled with a biconjugate gradient method to solve linear systems arising from the Newton algorithm. We provide three tests to show the effectiveness of the *nonlinear CVFE scheme* (2.7), (2.14). For these tests, we consider the following data: $L_\mathbf{x}=1$, $L_\mathbf{y}=1$ (the length and the width of the domain). We fix: $\Delta t = 0.002$, $\alpha = 0.01$, $\beta = 0.05$, $a\left(u\right) = d_u u\left(1-u\right)$, $d_u = 0.0005$, $\chi\left(u\right) = \zeta\times\left(u\left(1-u\right)\right)^2$, $\zeta = 0.05$. By definition, we have $\mu\left(u\right) = \frac{\zeta}{d_u}u\left(1-u\right)$ then, the numerical flux function μ_{KL}^{n+1} is given using the following functions:

$$\mu_\uparrow\left(z\right) = \mu\left(\min\{z,\frac{1}{2}\}\right), \quad \text{and } \mu_\downarrow\left(z\right) = \mu\left(\max\{z,\frac{1}{2}\}\right) - \mu\left(\frac{1}{2}\right), \quad \forall z\in\left(0,1\right)\times\left(0,1\right).$$

Unless stated otherwise and throughout the tests, we assume that $f\left(u\right)=0$, that the initial conditions are defined by regions, and we assume zero-flux boundary conditions. For instance, the cell density is initially defined by $u_0\left(\mathbf{x},\mathbf{y}\right)=1$ in the square region given by $\left(\mathbf{x},\mathbf{y}\right)\in[0.45,\,0.55]$ and 0 otherwise. The initial chemeoattractant concentration is defined by $v_0\left(\mathbf{x},\mathbf{y}\right)=5$ in the space region given by $\left(\mathbf{x},\mathbf{y}\right)\in[0.2,\,0.3]\times[0.45,\,0.55]\cup[0.45,\,0.55]\times[0.2,\,0.3]\cup[0.45,\,0.55]\times[0.7,\,0.8]\cup[0.7,\,0.8]\times[0.45,\,0.55]$.

Test 1 (Weak anisotropic case). In this test, we assume that the diffusion tensors are given by

$$\Lambda(\mathbf{x}) = \begin{pmatrix}1 & 0\\ 0 & \theta\end{pmatrix}, \quad D(\mathbf{x}) = d\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad d = 0.0001.$$

Further, we consider an admissible triangular primary mesh made of 14 336 triangles, the corresponding Donald dual mesh consists of 7 297 dual control volumes. In a admissible triangular mesh, all the angles of triangles are acute, then one can deduce that the maximum principle is verified for v since the transmissibility coefficients are nonnegative, which it is not the case for u. In Tab. 6.1, we present minimum and maximum values obtained with each of the scheme (2.7)–(2.8), the *nonlinear CVFE scheme* (2.7),(2.14), and the finite volume scheme.

		scheme (2.7)–(2.8)	scheme (2.7),(2.14)	FV scheme
After 1 iteration	Min. Val. u	0.0	0.0	0.0
$\theta = 1$	Max. Val. u	1.0	1.0	1.0
After 10 iterations	Min. Val. u	0.0	0.0	0.0
$\theta = 1$	Max. Val. u	0.971110	1.0	1.0
After 1 iteration	Min. Val. u	-1.73001×10^{-3}	8.68789×10^{-20}	$\vert$
$\theta = 5$	Max. Val. u	0.99722922	1.0	$\vert$
After 10 iterations	Min. Val. u	-1.62500×10^{-2}	0.00	$\vert$
$\theta = 5$	Max. Val. u	0.9715705	1.0	$\vert$
After 1 iteration	Min. Val. u	-4.46953×10^{-3}	6.30555×10^{-16}	$\vert$
$\theta = 10$	Max. Val. u	1.00018368	1.0	$\vert$
After 10 iterations	Min. Val. u	-3.91245×10^{-2}	6.30554×10^{-16}	$\vert$
$\theta = 10$	Max. Val. u	0.98342428	0.9999999	$\vert$

TABLE 6.1. Numerical results after 1 and 10 iterations.

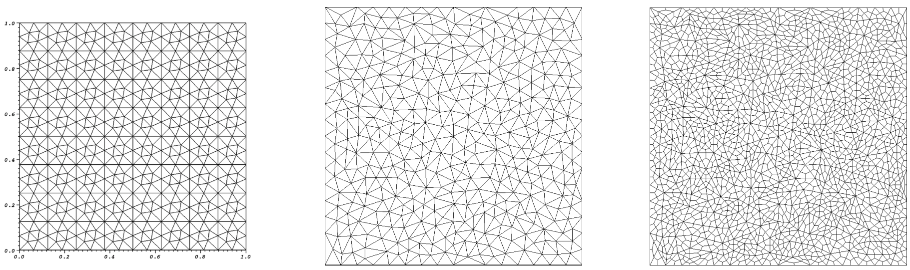

FIGURE 6.1. Meshes: admissible mesh for Test 1(left), initial primal mesh for Test 2 and 3 (center) and barycentric dual mesh for Test 2 and 3 (right).

Test 2 (Weak anisotropic case/obtuse angles). In this test, we consider a general unstructured mesh that contains obtuse angles, this mesh is made of 5 193 triangles and 2 665 dual control volumes. The discrete maximum principle is not guarantied for v, hence we cannot expect the maximum principle for u since the computation of u depends on the values of v, for that we consider the nonlinear discretization (2.14) of v.

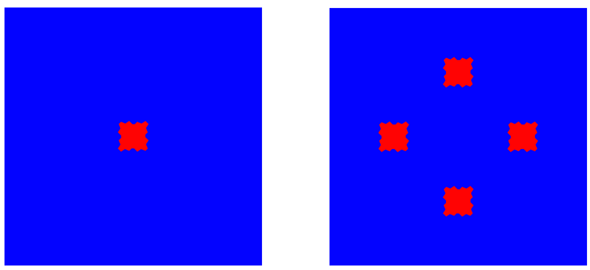

FIGURE 6.2. Initial condition for the cell density u (left) with $0 \leq u \leq 1$ and for the chemeoattractant concentration v (right) with $0 \leq v \leq 5$.

The diffusion tensors are defined, for all $\mathbf{x} \in (0,1)\times(0,1)$, by

$$\Lambda(\mathbf{x}) = \begin{pmatrix} 7 & 2 \\ 2 & 10 \end{pmatrix}, \quad D(\mathbf{x}) = d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = 0.0001.$$

Figure 6.2 represents initial distributions of the cell density u and the chemeoattractant concentration v over the initial triangular mesh as well as the corresponding dual mesh.

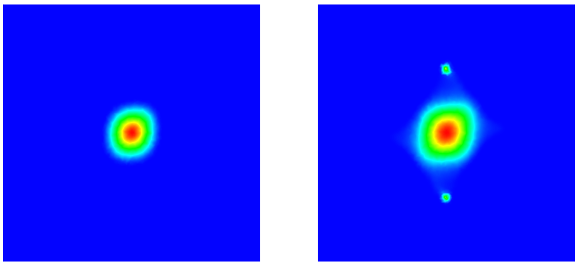

Figure 6.3. Evolution of the cell density u at time $t = 0.4$ with $0 \leq u \leq 0.667$ (left), and at time $t = 1.4$ with $0 \leq u \leq 0.632$(right).

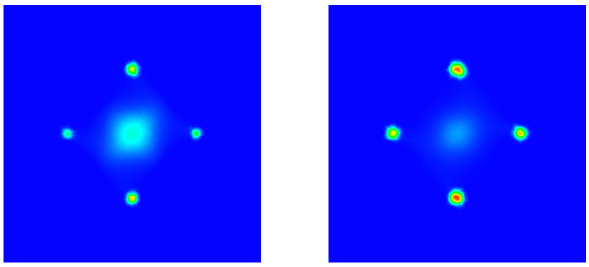

Figure 6.4. Evolution of the cell density u at time $t = 2.4$ with $0 \leq u \leq 0.972$ (left), and at time $t = 4$ with $0 \leq u \leq 0.987$(right).

Figures 6.3–6.4 represent the evolution of the cell density at time $t = 0.4$, $t = 1.4$, $t = 2.4$, and $t = 4$. At moment $t = 0.4$, it is clear that the cell density diffuses in the space without any interactions with the chemeoattractant which diffuses uniformly in the space. Then, after a while, and when the chemeoattractant diffusion reaches the cell density location, we see that the latter changes its direction to be absorbed by the chemeoattractant located vertically. This process continues and the cells accumulate into the location of the chemeoattractant and finally we obtain the cell density aggregations as shown at $t = 4$.

Test 3 (Anisotropic case/obtuse angles). In this test, we consider an unstructured mesh consisting of 15 568 primal triangles and 7 912 dual control dual volumes. Further, we assume that the diffusion tensors are anisotropic and are given by:

$$\Lambda(\mathbf{x}) = \begin{pmatrix} 8 & -7 \\ -7 & 20 \end{pmatrix}, \quad D(\mathbf{x}) = d\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad d = 0.0001.$$

Table 6.2 provides a comparison between the *nonlinear CVFE scheme* coupled on the one hand with the discretization (2.8) of v and with the discretization (2.14) of v on the other hand. We see that the discretization (2.14) carries out a better approximation than the discretization (2.8) in terms of ensuring the discrete maximum principle property.

		CVFE scheme: (2.7)–(2.8)	*CVFE scheme:* (2.7),(2.14)
After 1 iteration	Min. Val. u	0.0	0.0
	Max. Val. u	1.0	1.0
	Min. Val. v	-1.141912E-002	1.922764E-051
	Max. Val. v	5.012383	4.999982
After 200 iterations	Min. Val. u	0.0	0.0
	Max. Val. u	0.5298226	0.5312562
	Min. Val. v	-1.731068E-003	1.297192E-080
	Max. Val. v	4.8053827	4.8018742
After 1000 iterations	Min. Val. u	0.0	0.0
	Max. Val. u	0.9957580	0.9974757
	Min. Val. v	6.265859E-023	3.171769E-080
	Max. Val. v	2.961761	2.910828

Table 6.2. Numerical results after 1, 200 and 1000 iterations over an unstructured mesh with obtuse angles.

References

[1] M. Afif and B. Amaziane. Convergence of finite volume schemes for a degenerate convection–diffusion equation arising in flow in porous media. *Computer methods in applied mechanics and engineering*, 191(46):5265–5286, 2002.

[2] A. Agouzal, J. Baranger, J-F Maitre, and F. Oudin. Connection between finite volume and mixed finite element methods for a diffusion problem with nonconstant coefficients. Application to a convection diffusion problem. *EAST WEST J NUMER MATH*, 3(4):237–254, 1995.

[3] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.

[4] B. Andreianov, M. Bendahmane, and M. Saad. Finite volume methods for degenerate chemotaxis model. *Journal of computational and applied mathematics*, 235(14):4015–4031, 2011.

[5] B. Andreianov, C. Cancès, and A. Moussa. A nonlinear time compactness result and applications to discretization of degenerate parabolic-elliptic PDEs. https://hal.archives-ouvertes.fr/hal-01142499/document, 2015.

[6] K. Brenner and R. Masson. Convergence of a vertex centered discretization of two-phase darcy flows on general meshes. *International Journal of Finite Volume*, 10:1–37, 2013.

[7] Zhiqiang Cai. On the finite volume element method. *Numerische Mathematik*, 58(1):713–735, 1990.

[8] C. Cancès and C. Guichard. Entropy-diminishing CVFE scheme for solving anisotropic degenerate diffusion equations. *Finite Volumes for Complex Applications VII-Methods and Theoretical Aspects*, pages 187–196, 2014.

[9] C. Cancès and C. Guichard. Convergence of a nonlinear entropy diminishing control volume finite element scheme for solving anisotropic degenerate parabolic equations. *Math. Comp.*, 85(298):549–580, 2016.

[10] C. Cancès and C. Guichard. Numerical analysis of a robust free energy diminishing finite volume scheme for parabolic equations with gradient structure. *Found. Comput. Math.*, pages 1–60, 2016.

[11] Cathala M. Cancès C. and Le Potier C. Monotone corrections for generic cell-centered finite volume approximations of anisotropic diffusion equations. *Numerische Mathematik*, 3:387–417, 2013.

[12] G. Chamoun, M. Saad, and R. Talhouk. Monotone combined edge finite volume–finite element scheme for anisotropic keller–segel model. *Numerical Methods for Partial Differential Equations*, 30(3):1030–1065, 2014.

[13] G. Chavent, J. Jaffré, and JE Roberts. Mixed-hybrid finite elements and cell-centred finite volumes for two-phase flow in porous media. *Mathematical Modelling of Flow Through Porous Media*, pages 100–114, 1995.

[14] Y. Coudière, J-P. Vila, and P. Villedieu. Convergence rate of a finite volume scheme for a two dimensional convection-diffusion problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(03):493–516, 1999.

[15] J. Droniou. Finite volume schemes for diffusion equations: introduction to and review of modern methods. *Mathematical Models and Methods in Applied Sciences*, 24(08):1575–1619, 2014.

[16] J. Droniou and R. Eymard. Uniform-in-time convergence of numerical methods for non-linear degenerate parabolic equations. *Numer. Math.*, 132(4):721–766, 2016.

[17] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, and R. Herbin. The gradient discretisation method . https://hal.archives-ouvertes.fr/hal-01382358, November 2016.

[18] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations. *Mathematical Models and Methods in Applied Sciences*, 23(13):2395–2432, 2013.

[19] R. Eymard, T. Gallouët, M. Ghilani, and R. Herbin. Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes. *IMA J. Numer. Anal.*, 18(4):563–594, 1998.

[20] R. Eymard, T. Gallouët, and A. Herbin, R. qnd Michel. Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numerische Mathematik*, 92(1):41–82, 2002.

[21] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. *Handbook of numerical analysis*, 7:713–1018, 2000.

[22] R. Eymard, T. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes sushi: a scheme using stabilization and hybrid interfaces. *IMA Journal of Numerical Analysis*, 30(4):1009–1043, 2010.

[23] T. Gallouët. Some discrete functional analysis tools. In C. Cancès and P. Omnes, editors, *Finite Volumes for Complex Applications VIII*, Proceedings in Mathematics and Statistics. Springer, 2017.

[24] E. Godlewski and P.A. Raviart. *Hyperbolic systems of conservation laws*, volume 3–4 of *Mathematics and Applications*. Ellipses, Paris, 1991.

[25] M. Ibrahim and M. Saad. On the efficacy of a control volume finite element method for the capture of patterns for a volume-filling chemotaxis model. *Computers & Mathematics with Applications*, 68(9):1032 – 1051, 2014.

[26] E. F. Keller and L. A. Segel. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology*, 26(3):399–415, 1970.

[27] E. F. Keller and L. A. Segel. Model for chemotaxis. *Journal of Theoretical Biology*, 30(2):225–234, 1971.

[28] R.J LeVeque. Nonlinear conservation laws and finite volume methods. In *Computational methods for astrophysical fluid flow*, pages 1–159. Springer, 1998.