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Erratum to : "A family of second-order dissipative finite volume schemes for hyperbolic systems of conservation laws"

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Abstract. We, authors of the paper entitled "A family of second-order dissipative finite volume schemes for hyperbolic systems of conservations laws" present our apologies for many mistakes in the current version of the published paper. We explain below the mathematical mistakes and how to modify the assumption of the Theorem (4.2) to get the same conclusion (namely, the decay of the mass of the entropy). New material and a correct proof of the Theorem (4.2) are given. A corrected version of the published paper has been uploaded on HAL server at the adress: `https://hal.science/hal-03564325`.

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1. Issues in the published version

The mathematical errors are located in Section 4 (Global entropy inequality in the general case). In particular, Lemma 4.3 (Poincaré inequality) is wrong. It asserts a result of monotony of matrices (see the second point in the section below) which is wrong for non diagonal matrices. Moreover, since in Section 4 we work with a general convex set Ω of R^d, if $0 \quad \Omega$, there is a mathematical contradiction with the fact that we consider in the assumption of the Theorem 4.2 compactly supported state $(w_i^n) \quad \Omega$. What misleads us, is the linearized analysis of Section 3, which has to be understood as a stability analysis around a constant state in Ω. Actually, in Section 3, (w_i^n) is rather understood as a fluctuation and this is why the $l^2(\mathsf{Z})$ space is adapted.

2. The wrong arguments

The Lemma (4.3) which was intended to prove a Poincaré inequality is wrong. It relies on two points:

(1) It is supposed that the sequence $(w_i^n)_{i \ \mathsf{Z}}$ is compactly supported. But provided $0 \ / \ \Omega$, it yields a contradiction with the fact that we also ask $(w_i^n)_{i \ \mathsf{Z}} \quad \Omega$. The argument was intended to mimick the case of the linearized equations around a constant state which misleads us and we do apologize for this.

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(2) The last point but not the least: it uses at the beginning of the proof a fake argument which is the following : let $A \quad M_d(\mathsf{R})$ a symmetric positive definite matrix, then there exists $C > 0$ such that for all $x, y \quad \mathsf{R}^d$, $\sum_{j=1}^{d} |(Ax)_j||y_j| \geq C \sum_{j=1}^{d} |x_j||y_j|$. It is wrong for non diagonal matrices as soon as $d > 1$, as we prove here under.

Lemma 2.1. *Let $d > 1$ and $A = (a_{ij}) \quad M_d(\mathsf{R})$ a non diagonal symmetric positive definite matrix. Then for all $C > 0$, there exists $x, y \quad \mathsf{R}^d$ such that $\sum_{j=1}^{d} |(Ax)_j||y_j| < C \sum_{j=1}^{d} |x_j||y_j|$.*

Proof. Let $C > 0$. Since A is positive definite, one has for all $i \quad \{1, \dots, d\}$, $a_{ii} > 0$ and since it is non diagonal there exists a couple $(i \;, j \;) \quad \{1, \dots, d\}^2$ with $i \; = j$ such that $a_{i^* j^*} = 0$. So consider $y = e_{i^*}$ the canonical basis vector of R^d. Let $x = (x_1, \dots, x_{i^*-1}, 1, x_{i^*+1}, \dots, x_d)^T \quad \mathsf{R}^d$ belonging to the non-empty hyperplane

$$P_{i^*} = \{w \quad \mathsf{R}^d \; : \; a_{i^*} \cdot w = 0\},$$

where $a_i = (a_{i^*1}, \dots, a_{i^*d})^T = 0$. Then one has, one the one hand $\sum_{j=1}^{d} |(Ax)_j||y_j| = |(Ax)_{i^*}| = 0$ because $x \quad P_{i^*}$ and on the other hand $\sum_{j=1}^{d} |x_j||y_j| = |x_{i^*}| = 1$. Hence, $\sum_{j=1}^{d} |(Ax)_j||y_j| < C \sum_{j=1}^{d} |x_j||y_j|$. ■

3. Erratum

Mathematical mistakes are in Section 4. We provide here after the corrections to consider in order to get the conclusion of the Theorem (4.2) still valid.

3.1. List of points to adress

(1) About the convergence of the various series:

(a) For the stability analysis, to justify that the various series are convergent, what is precisely needed is the control in l^2 of the discrete derivatives. So the adapted framework is the homogeneous Sobolev space defined by

$$\dot{h}^2(\mathsf{Z}; \Omega) := \left\{ w \quad \Omega^{\mathsf{Z}} \; : \; (w_{i+1} - w_i)_{i \; \mathsf{Z}} \quad l^2(\mathsf{Z}), \; (w_{i+1} - 2w_i + w_{i-1})_{i \; \mathsf{Z}} \quad l^2(\mathsf{Z}) \right\} \tag{3.1}$$

which is no longer in contradiction with the fact that one may consider constante state for $(w_i^n) \quad \Omega$.

(b) The convergence in l^2 of the discrete derivatives alone is not sufficient. One also needs that the map $s \quad [0,1] \to Q_i^n(s)$, $s \quad [0,1] \to F_i^n(s)$ and $(u,s) \quad [0,1]^2 \to N_i(us)$ given in (4.5), (4.7) and (4.11) are bounded uniformly for $s \quad [0,1]$ and $i \quad \mathsf{Z}$. Noticing that in the definition of these matrices the argument is a convex combination of two elements, we have no choice that assuming that $(w_i^n) \quad K^n$ where $K^n \quad \Omega$ is a convex compact subset. We thus need the following lemma.

Lemma 3.1. *Let Ω a non empty convex open set. Let $K \quad \Omega$ a convex compact subset. Let a, b two elements in K. If $A : \Omega \to M_d(\mathsf{R})$ is a continuous map then for any two points $a, b \quad K$ the map $s \quad [0,1] \to A((1-s)a + sb)$ is continuous on $[0,1]$. In addition, for any matrix norm $\; \cdot \;$:*

$$\sup_{s \; [0,1]} \; A((1-s)a + sb) \; \leq \max_{w \; K} \; A(w) \; .$$

Proof. Since $a, b \quad K$ and K is a convex set one has $(1 - s)a + sb \quad K$ for all $s \quad [0, 1]$. Therefore the map $A : s \quad [0, 1] \to \quad A((1 - s)a + sb)$ is well-defined and continuous on $[0, 1]$ by composition of continuous maps. Since $[0, 1]$ is a compact set in R, there exists $s \quad [0, 1]$ such that $\sup_{s \; [0,1]} \; A((1 - s)a + sb) = \; A((1 - s \;)a + s \; b)$. Then since $(1 - s \;)a + s \; b \quad K$ one has also $\; A((1 - s \;)a + s \; b) \leq \sup_{w \; K} \; A(w) \; = \max_{w \; K} \; A(w)$ where the supremum is a maximum because K is a compact set. ∎

(2) In Proposition (4.4) one has to select the parameter θ such that the inequality (4.4) holds. Specifically, one needs to justify that the denominator in (4.11) cannot be zero. To remedy this, we precisely eliminate this case. So it leads us to consider,

$$S_\eta(\mathsf{Z}; \Omega) := \left\{ w \quad \dot{h}^2(\mathsf{Z}; \Omega) \; : \sum_{j=1}^{d} \sum_{i \; \mathsf{Z}} |(\quad \eta(w_{i+1}) - \quad \eta(w_{i-1}))_j||(w_{i+1} - 2w_i + w_{i-1}))_j| = 0 \right\}. \quad (3.2)$$

Note that $S_\eta(\mathsf{Z}; \Omega)$ does not contain constante state. However it is not a restriction because in this case, taking (Θ_i^n) to be zero for example, the scheme (4.1) is exact (because it is conservative).

3.2. Correction of the statements

- In the introduction, the sentence at the end of page 2 must be completed with the word convex. It should be written: "Namely, it is required to belong to a convex compact subset of Ω."

- In the Definition 2.3, the third sentence must be completed with the word strictly. It should be written : Let $\eta \quad C^1(\Omega, \mathsf{R})$ be a strictly convex entropy function.

- In Section 3, the second sentence should be replaced by: We therefore consider momentarily a linearization of a non linear hyperbolic scalar equation around a constant state which here takes the form

$$\begin{cases} \partial_t w + a\partial_x w &= 0, \quad t > 0, \quad x \quad \mathsf{R}, \\ w(x, t = 0) &= w_0(x) \end{cases} \quad (3.3)$$

where $a = 0$.

- The correct statement of Theorem 4.3 is the following:

Theorem 3.2 (Global entropy inequality). *Let Ω be a non-empty convex open subset of R^d. Consider $(\eta, G) \quad C^2(\Omega, \mathsf{R}) \times C^1(\Omega, \mathsf{R})$ a pair of strictly convex entropy and entropy-flux which satisfies (1.2). Let $(w_i^n)_{i \; \mathsf{Z}} \quad S_\eta(\mathsf{Z}; \Omega)$ such that $\sum_{i \; \mathsf{Z}} \eta(w_i^n)\Delta x$ is finite. Let $(\Theta_i^n)_{i \; \mathsf{Z}}$, a sequence of bounded matrices such that*

$$\mathsf{S}^n := \int_0^1 \sum_{i \; \mathsf{Z}} N_i^n(s) P_i^n(s) D_i^n \cdot D_i^n \mathrm{d}s > 0, \quad (3.4)$$

where the block matrices (D_i^n, N_i^n, P_i^n) $\mathrm{R}^{2d} \times (M_{2d}(\mathrm{R}))^2$ *are respectively defined by*

$$
\begin{aligned}
D_i^n &= \begin{pmatrix} \delta^n_{i-\frac{1}{2}} \\ \delta^n_{i+\frac{1}{2}} \end{pmatrix}, \\
N_i^n(s) &= \begin{pmatrix} {}^2\eta\left(w_i^n - s\delta^n_{i-\frac{1}{2}}\right) & 0 \\ 0 & {}^2\eta\left(w_i^n + s\delta^n_{i+\frac{1}{2}}\right) \end{pmatrix}, \\
P_i^n(s) &= \begin{pmatrix} (1-2s)I + \Theta_i^n & -\Theta_i^n \\ \Theta_i^n - I & 2(1-s)I - \Theta_i^n \end{pmatrix},
\end{aligned}
\tag{3.5}
$$

for all s $[0,1]$*. Also assume there exists a convex compact set* K^n Ω *such that* $(w_i^n)_{i \ \mathrm{Z}}$ K^n*. Let the numerical di usion* λ *be such that*

$$\lambda > \lambda^n, \tag{3.6}$$

where

$$
\begin{aligned}
\lambda^n &= \frac{2\max\left(0, \sum_{i \ \mathrm{Z}} \int_0^1 s\left(\int_0^1 N_i^n(us)\mathrm{d}u\right) F_i^n(s) D_i^n \cdot D_i^n \mathrm{d}s\right)}{\sum_{i \ \mathrm{Z}} \int_0^1 N_i^n(s) P_i^n(s) D_i^n \cdot D_i^n \mathrm{d}s} \geq 0, \\
F_i^n(s) &= \begin{pmatrix} - \ f\left(w_i^n - s\delta^n_{i-\frac{1}{2}}\right) & 0 \\ 0 & f\left(w_i^n + s\delta^n_{i+\frac{1}{2}}\right) \end{pmatrix}, \quad s \ \ [0,1],
\end{aligned}
\tag{3.7}
$$

and $\frac{\Delta t}{\Delta x}$ *be such that*

$$0 < \frac{\Delta t}{\Delta x} \leq \frac{-\sum_{i \ \mathrm{Z}} \ \eta(w_i^n) \cdot R_i^n}{\int_0^1 (1-s) \sum_{i \ \mathrm{Z}} \ {}^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x} R_i^n\right) R_i^n \cdot R_i^n \mathrm{d}s}. \tag{3.8}$$

If $\Omega = \mathrm{R}^d$*, then one has the global entropy inequality,*

$$\sum_{i \ \mathrm{Z}} \eta(w_i^{n+1})\Delta x \leq \sum_{i \ \mathrm{Z}} \eta(w_i^n)\Delta x. \tag{3.9}$$

If $\Omega = \mathrm{R}^d$ *then the global entropy inequality* (3.9) *still holds if moreover* $0 < \frac{\Delta t}{\Delta x} \leq c^n$ *where* c^n *is a positive non explicit constant given by the Lemma (4.1).*

- Lemma 4.3 (Poincaré inequality) should be removed.
- The conclusion of Proposition 4.3 (Existence of dissipative corrections) is still valid but it needs supplementary material (see the Section 3.3 below).
- In the Section (4.1) entitled “Reformulation of the global dissipation”: at the begining of the section, it should be mentionned that owing to Propositions 3.4 and 3.5 (below), it is fully understood, in this section, that the various series are convergent so that we only concentrate on algebraic computations.

- In accordance with the previous point, in the Section (4.1), for the series to be convergent, one must replace, in Lemma (4.5) and (4.6), the assumptions by the following ones: Let the sequence $(w_i^n)_{i\in\mathbb{Z}} \in \mathcal{S}_\eta(\mathbb{Z};\Omega)$ and let $(\eta, G) \in C^2(\Omega,\mathbb{R}) \times C^1(\Omega,\mathbb{R})$ a pair of strictly convex entropy-entropy-flux which satisfies (1.2). Assume in addition that there exists a convex compact subset $K^n \subset \Omega$ such that $(w_i^n) \subset K^n$.

3.3. New material to complete the Proof of Theorem 3.2

We give below the new material to complete the proof of the Theorem 3.2.

3.3.1. *Preparatory material*

Before proving the main theorem, we need preparatory material. It is needed to justify the convergence of the various series and specifically to justify the finiteness of the numerical diffusion (3.7).

Lemma 3.3 (Relative uniform upper bound). *Let Ω a non empty convex open set. Let $K \subset \Omega$ a convex compact set. Let a, b two elements in K. If $A : \Omega \to \mathcal{M}_d(\mathbb{R})$ is a continuous map then for any two points $a, b \in K$ the map $s \in [0,1] \to A((1-s)a + sb)$ is continuous on $[0,1]$. In addition, for any matrix norm $\|\cdot\|$:*

$$\sup_{s\in[0,1]} \|A((1-s)a + sb)\| \leq \max_{w\in K} \|A(w)\|.$$

Proof. Since $a, b \in K$ and K is a convex set one has $(1-s)a + sb \in K \subset \Omega$ for all $s \in [0,1]$. Therefore the map $A : s \in [0,1] \to A((1-s)a + sb)$ is well-defined and continuous on $[0,1]$ by composition of continuous maps. Since $[0,1]$ is a compact set in $\mathbb{R}$, there exists $s^\star \in [0,1]$ such that $\sup_{s\in[0,1]} \|A((1-s)a + sb)\| = \|A((1-s^\star)a + s^\star b)\|$. Then since $(1-s^\star)a + s^\star b \in K$ one has also $\|A((1-s^\star)a + s^\star b)\| \leq \sup_{w\in K} \|A(w)\| = \max_{w\in K} \|A(w)\|$ where the supremum is a maximum because K is a compact set. ∎

Proposition 3.4 (Existence of dissipative corrections). *Let $(w_i^n)_{i\in\mathbb{Z}} \in \mathcal{S}_\eta(\mathbb{Z};\Omega)$ and let $\eta \in C^2(\Omega,\mathbb{R})$ a strictly convex entropy. Assume in addition that there exists a convex compact subset $K^n \subset \Omega$ such that $(w_i^n) \subset K^n$. Then if $(\Theta_i^n)_{i\in\mathbb{Z}}$ verifies*

$$\Theta_i^n = -\theta\, \mathrm{diag}_{1\leq j\leq d}\left(\mathrm{sign}\left(\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)_j (\delta_{i+\frac12}^n - \delta_{i-\frac12}^n)_j\right)\right), \tag{3.10}$$

with

$$\theta > \frac{-\min\left(0, \displaystyle\int_0^1 \sum_{i\in\mathbb{Z}} Q_i^n(s) D_i^n \cdot D_i^n \mathrm{d}s\right)}{\displaystyle\sum_{\substack{i\in\mathbb{Z}\\ j\in\{1,\ldots,d\}}} \left|\left(\nabla\eta(w_{i+1}^n) - \nabla\eta(w_{i-1}^n)\right)_j (\delta_{i+\frac12}^n - \delta_{i-\frac12}^n)_j\right|}, \tag{3.11}$$

$$Q_i^n(s) = \begin{pmatrix} (1-2s)\nabla^2\eta(w_i^n - s\delta_{i-\frac12}^n) & -\nabla^2\eta(w_i^n + s\delta_{i+\frac12}^n) \\ 0 & 2(1-s)\nabla^2\eta(w_i^n + s\delta_{i+\frac12}^n) \end{pmatrix},$$

for all $s \in [0,1]$, then the dissipative inequality (3.4) holds.

Proof. We firstly prove that the right hand side in (3.11) is finite. By assumption the denominator in (3.11) cannot be zero. Moreover, one has using a Cauchy–Schwarz inequality

$$\left|\int_0^1 \sum_{i\in\mathbb{Z}} Q_i^n(s) D_i^n \cdot D_i^n \mathrm{d}s\right| \leq \sum_{i\in\mathbb{Z}} \int_0^1 \|Q_i^n(s)\|_2 \mathrm{d}s\, \|D_i^n\|_2^2 \leq \sum_{i\in\mathbb{Z}} \sup_{s\in[0,1]} \|Q_i(s)\|_2 \|D_i^n\|_2^2.$$

where $\|\cdot\|_2$ denotes both the Euclidean norm on R^{2d} and the induced norm on $M_{2d}(\mathsf{R})$. Using the equivalence of norm in a finite dimensional vector space one has in particular that there exists a constant $c(d)$ which only depends on the dimension d such that

$$\sup_{s\in[0,1]}\|Q_i(s)\|_2 \leq c(d)\sup_{s\in[0,1]}\|Q_i(s)\|_\infty .$$

From now, observe that thanks to Lemma 3.3 applied with $A=\nabla^2\eta$ and the matrix norm $\|\cdot\|_\infty$, we obtain for all $i\in\mathsf{Z}$

$$\sup_{s\in[0,1]}\|\nabla^2\eta((1-s)w_i+sw_{i-1})\|_\infty \leq \max_{w\in K^n}\|\nabla^2\eta(w)\|_\infty .$$

We therefore have that for all $i\in\mathsf{Z}$, $\sup\limits_{s\in[0,1]}\|Q_i(s)\|_\infty \leq 2\max\limits_{w\in K^n}\|\nabla^2\eta(w)\|_\infty$. It eventually yields the upper bound

$$\sum_{i\in\mathsf{Z}}\sup_{s\in[0,1]}\|Q_i(s)\|_2\|D_i^n\|_2^2 \leq 2c(d)\max_{w\in K^n}\|\nabla^2\eta(w)\|_\infty\sum_{i\in\mathsf{Z}}\|D_i^n\|_2^2<+\infty .$$

The last series being convergent because $(w_i^n)\in\dot{h}^2(\mathsf{Z};\Omega)$. So the ratio in (3.11) is finite. Next, since the matrix $(\Theta_i^n)_{i\in\mathsf{Z}}$ defined by (3.10) are symmetric (because they are diagonal) we have $\Theta_i^n a\cdot b=a\cdot\Theta_i^n b$, for all vectors $(a,b)\in(\mathsf{R}^d)^2$. As a consequence, from the definition of the matrices P_i^n, N_i^n given by (3.5) and the definition of S^n given in (3.4), we have

$$\mathsf{S}^n=\int_0^1\sum_{i\in\mathsf{Z}}Q_i^n(s)D_i^n\cdot D_i^n\mathrm{d}s-\sum_{i\in\mathsf{Z}}\Theta_i^n\left(\nabla\eta(w_{i+1}^n)-\nabla\eta(w_{i-1}^n)\right)\cdot\left(\delta^n_{i+\frac{1}{2}}-\delta^n_{i-\frac{1}{2}}\right).$$

Using the $(\Theta_i^n)_{i\in\mathsf{Z}}$ formula (3.10), we eventually obtain

$$\mathsf{S}^n=\int_0^1\sum_{i\in\mathsf{Z}}Q_i^n(s)D_i^n\cdot D_i^n\mathrm{d}s+\theta\sum_{\substack{i\in\mathsf{Z}\\ j\in\{1,\ldots,d\}}}\left|\left((\nabla\eta(w_{i+1}^n)-(\nabla\eta(w_{i-1}^n))\right)_j(\delta^n_{i+\frac{1}{2}}-\delta^n_{i-\frac{1}{2}})_j\right|,$$

which is positive with θ verifying the inequality (3.11). ∎

Proposition 3.5 (Finiteness of the numerical diffusion). *Let $(w_i^n)_{i\in\mathsf{Z}}\in\mathsf{S}_\eta(\mathsf{Z};\Omega)$ and let $\eta\in C^2(\Omega,\mathsf{R})$ a strictly convex entropy. Assume in addition that there exists a convex compact subset $K^n\subset\Omega$ such that $(w_i^n)\in K^n$. Then the numerical diffusion λ^n defined by (3.7) is positive and finite.*

Proof. We prove that the numerator in (3.7) is finite. The denominator was proven to be positive by construction of the matrix parameter $(\Theta_i^n)_{i\in\mathsf{Z}}$ in Proposition 3.4. So one has, using a Cauchy–Schwarz inequality and the fact that a matrix norm is submultiplicative

$$\left|\sum_{i\in\mathsf{Z}}\int_0^1 s\left(\int_0^1 N_i^n(us)\mathrm{d}u\right)F_i^n(s)D_i^n\cdot D_i^n\mathrm{d}s\right|\leq\sum_{i\in\mathsf{Z}}\sup_{s\in[0,1]}\|N_i^n(s)\|_2\sup_{s\in[0,1]}\|F_i^n(s)\|_2\|D_i^n\|_2^2$$

where $\|\cdot\|_2$ denotes both the Euclidean norm on R^{2d} and the induced norm on $M_{2d}(\mathsf{R})$. Using the equivalence of norm in a finite dimensional vector space one has in particular that there exist a constant $c(d)$ which only depends on the dimension d such that

$$\sup_{s\in[0,1]}\|N_i^n(s)\|_2\leq c(d)\sup_{s\in[0,1]}\|N_i^n(s)\|_\infty,\quad \sup_{s\in[0,1]}\|F_i^n(s)\|_2\leq c(d)\sup_{s\in[0,1]}\|F_i^n(s)\|_\infty .$$

Observe that thanks to Lemma 3.3 applied respectively with $A=\nabla^2\eta$ and $A=\nabla f$ and the matrix norm $\|\cdot\|_\infty$, we obtain for all $i\in\mathsf{Z}$

$$\sup_{s\in[0,1]}\|\nabla^2\eta((1-s)w_i+sw_{i-1})\|_\infty\leq\max_{w\in K^n}\|\nabla^2\eta(w)\|_\infty,$$
$$\sup_{s\in[0,1]}\|\nabla f((1-s)w_i+sw_{i-1})\|_\infty\leq\max_{w\in K^n}\|\nabla f(w)\|_\infty .$$

It therefore yields that for all $i \quad \mathsf{Z}$,

$$\sup_{s \quad [0,1]} \quad N_i^n(s) \quad {}_\infty \leq \max_{w \quad K^n} \quad {}^2\eta(w) \quad {}_\infty, \qquad \sup_{s \quad [0,1]} \quad F_i^n(s) \quad {}_\infty \leq \max_{w \quad K^n} \quad f(w) \quad {}_\infty.$$

So we eventually obtain,

$$\sum_{i \quad \mathsf{Z}} \sup_{s \quad [0,1]} \quad N_i^n(s) \quad {}_2 \sup_{s \quad [0,1]} \quad F_i^n(s) \quad {}_2 \quad D_i^n \quad {}_2 \leq c(d)^2 \max_{w \quad K^n} \quad {}^2\eta(w) \quad {}_\infty \max_{w \quad K^n} \quad f(w) \quad {}_\infty \sum_{i \quad \mathsf{Z}} \quad D_i^n \quad {}_2^2 < +\infty.$$

The last series is finite because $(w_i^n) \quad \dot{h}^2(\mathsf{Z}; \Omega)$. ∎

3.3.2. *Proof of the Theorem 3.2 (related to the Section (4.2) "Proof of the main result")*

Let $(\eta, G) \quad C^2(\Omega, \mathsf{R}) \times C^1(\Omega, \mathsf{R})$ a pair of strictly convex entropy, entropy-flux which satisfies (1.2). Consider the sequence $(w_i^n)_{i \quad \mathsf{Z}}$ verifying the assumptions of the Theorem 3.2. Consider the CFL condition $\frac{\Delta t}{\Delta x}$ given in the Theorem 3.2. Therefore the sequence $(w_i^{n+1})_{i \quad \mathsf{Z}}$ is contained in Ω. Since $\eta \quad C^2(\Omega, \mathsf{R})$, using a Taylor expansion, in the above equations, we deduce

$$\eta(w_i^{n+1}) = \eta(w_i^n) + \frac{\Delta t}{\Delta x} \quad \eta(w_i^n) \cdot \mathsf{R}_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \int_0^1 (1-s) \quad {}^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathsf{R}_i^n\right) \mathsf{R}_i^n \cdot \mathsf{R}_i^n \mathrm{d}s.$$

It implies that,

$$\sum_{i \quad \mathsf{Z}} \eta(w_i^{n+1})\Delta x = \sum_{i \quad \mathsf{Z}} \eta(w_i^n)\Delta x + \frac{\Delta t}{\Delta x} \sum_{i \quad \mathsf{Z}} \quad \eta(w_i^n) \cdot \mathsf{R}_i^n \Delta x$$
$$+ \left(\frac{\Delta t}{\Delta x}\right)^2 \int_0^1 (1-s) \sum_{i \quad \mathsf{Z}} \quad {}^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathsf{R}_i^n\right) \mathsf{R}_i^n \cdot \mathsf{R}_i^n \mathrm{d}s \Delta x.$$

The second sum after the equality was proven being convergent thanks to Propositions 3.4 and 3.5 (of this document). We now justify that the third series inherited from the second order term in the Taylor expansion is finite. Owing to the Lemma (4.1), since $0 < \frac{\Delta t}{\Delta x} < c^n$, one has for all $s \quad [0,1]$, for all $i \quad \mathsf{Z}$,

$$w_i^n + s\frac{\Delta t}{\Delta x}\mathsf{R}_i^n \quad K^{n+1} := K^n + \overline{B}(0, \frac{\Delta t}{\Delta x}R^n) \quad \Omega$$

where $\overline{B}(0, \frac{\Delta t}{\Delta x}R^n)$ is the closed ball (for a norm $\quad \cdot \quad$ on R^d) centered at 0, of radius $\frac{\Delta t}{\Delta x}R^n$ and where $R^n > 0$ is a non explicit constant. Since K^{n+1} is the sum of two compact sets, then K^{n+1} is also a compact set. So repeating the same arguments as in Propositions 3.4 and 3.5 (of this document), one infers that there exists a constant $c(d)$ that only depends on the dimension d such that

$$\Big| \int_0^1 (1-s) \sum_{i \quad \mathsf{Z}} \quad {}^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathsf{R}_i^n\right) \mathsf{R}_i^n \cdot \mathsf{R}_i^n \mathrm{d}s \Big| \leq c(d) \max_{w \quad K^{n+1}} \quad {}^2\eta(w) \quad {}_\infty \sum_{i \quad \mathsf{Z}} \quad \mathsf{R}_i^n \quad {}_2^2 < +\infty.$$

Using again a similar argument as in Propositions 3.4 and 3.5 (that we do not detail), one again shows that the above series is finite because $(w_i^n) \quad \dot{h}^2(\mathsf{Z}; \Omega)$, the matrix parameter $(\Theta_i^n)_{i \quad \mathsf{Z}}$ is bounded and $(w_i^n) \quad K^n$. To establish the global entropy inequality (3.9), it is now sufficient to prove the following inequality

$$\sum_{i \quad \mathsf{Z}} \quad \eta(w_i^n) \cdot \mathsf{R}_i^n + \frac{\Delta t}{\Delta x} \int_0^1 (1-s) \sum_{i \quad \mathsf{Z}} \quad {}^2\eta\left(w_i^n + s\frac{\Delta t}{\Delta x}\mathsf{R}_i^n\right) \mathsf{R}_i^n \cdot \mathsf{R}_i^n \, \mathrm{d}s \leq 0. \tag{3.12}$$

Thanks to the Lemmas (4.6) and (4.7), we deduce that the dissipation reformulates as

$$\sum_{i \quad \mathsf{Z}} \quad \eta(w_i^n) \cdot \mathsf{R}_i^n = \sum_{i \quad \mathsf{Z}} \int_0^1 \left(\frac{s}{2}\left(\int_0^1 N_i^n(us)\mathrm{d}u\right) F_i^n(s) - \frac{\lambda}{4} N_i^n(s) P_i^n(s)\right) D_i^n \cdot D_i^n \mathrm{d}s. \tag{3.13}$$

But, as the sequence of matrices $(\Theta_i^n)_{i \in \mathbb{Z}}$ is selected in order to satisfy the inequality (3.4), and λ is such that the inequality (3.6) is verified, we have

$$\sum_{i \in \mathbb{Z}} \eta(w_i^n) \cdot \boldsymbol{R}_i^n < 0,$$

which is the definition of a dissipative flux given by (2.3). Using the CFL condition (3.8) concludes the proof.