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Convergence of a CVFE finite volume scheme for nonisothermal immiscible incompressible two-phase flow in porous media

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Abstract. We consider a two-phase Darcy flow in a heterogeneous porous medium while taking into account the effects of temperature. The set of governing equations consists of the usual equations derived from the mass conservation of both fluids along with the Darcy–Muskat, the capillary pressure laws, and the energy balance equation. The problem is written in terms of the fractional flow formulation, i.e. the saturation of one phase, the nonisothermal global pressure and the temperature are primary unknowns. The spatial and temporal discretizations are carried out applying a vertex-centered finite volume scheme and the implicit Euler scheme, respectively, resulting in the final system of fully coupled nonlinear equations. Under some realistic assumptions on the data, we establish a sufficient condition on the mesh to demonstrate the discrete maximum principle for saturation and temperature. Various a priori estimates are derived that yield an existence result for discrete solutions. Based on preliminary estimates and compactness arguments, we prove the convergence of the numerical scheme to a weak solution of the continuous model. The open source platform DuMu^X has been used to implement the resulting algorithm. Two numerical experiments are presented to illustrate the effectiveness and the robustness of the scheme.

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1. Introduction

Multiphase flows in porous media are involved in many applications related to reservoir simulation, subsurface environment and energy issues. We can mention non exhaustively production of geothermal energy, geological sequestration of gas (H_2, CO_2, CH_4) or nuclear waste management. Although different, these applications actually have many similarities. Finite volume schemes are the most commonly used methods for solving the systems modeling these problems. They are locally mass conservative schemes which is essential when solving such applications.

During recent decades, several finite volume discretizations have been developed for the resolution of coupled systems describing immiscible incompressible two-phase flows in porous media [2, 7, 14, 16, 17, 21, 22, 29]. Let us mention that in [2] and [29], nonlinear finite volume methods are discussed. In these

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works and the references therein, the authors were mainly interested in and addressed the difficulties raised in the isothermal setting, namely the construction of stable and consistent fluxes on irregular grids with anisotropy as well as the presence or absence of the discontinuous capillary pressure. They elaborated the key elements to establish the convergence analysis of the proposed numerical schemes. It turns out that the isothermal framework is not sufficient to handle the physics of some applications arising for instance in the simulation of geothermal energy production, high-level radioactive waste repositories or geological sequestration of gas. Incorporating thermal behavior is essential for such systems, and this work is motivated by that need. Despite some progress in the numerical simulation of two-phase problems under nonisothermal conditions, see for instance [3, 10, 20, 23, 28], the numerical analysis of these models is still lacking. To our best knowledge, the convergence analysis of finite volume schemes for nonisothermal two-phase flows in porous media are missed in the existing literature. Up to this point, there have been only one other recently published paper [5] on the subject. A convergence study was performed for the discretization of nonisothermal compressible two-phase flow in porous media using the cell-centered Two-Point Flow Approximation (TPFA) finite volume method. Concerning the mathematical analysis of such systems for nonisothermal flows, only recently few results have been obtained, for more details we refer to [6, 11, 12].

The aim of this paper is to investigate a two-phase model for heterogeneous porous media that takes into account the different reservoir temperatures to accurately capture flow physics. More precisely, we integrate temperature effects into immiscible incompressible two-phase flow in porous media. The basic equations for nonisothermal two-phase flow in a porous medium involve mass conservation, Darcy's law, energy conservation, saturation, and capillary pressure constraint equations. The governing fluid and heat transport equations used to model thermal recovery processes are highly nonlinear. As temperature variations influence fluid properties, and fluids transport heat as they move (convective flow), there is a strong coupling between the mass balance and energy balance equations. The problem is written in terms of the fractional flow formulation, i.e. the saturation of one phase, the nonisothermal global pressure [13] and the temperature are primary unknowns.

In this paper, we focus our attention on the study of nonisothermal immiscible, incompressible two-phase flow in heterogeneous porous media taking into account capillary effects using a fully coupled fully implicit finite volume scheme. We combine the advantages of the CVFE (control volume finite element) method to accurately solve the diffusion terms with an upwind method for space discretization on regular or unstructured grids. Time integration is based on implicit Euler scheme to allow large enough time steps. This work is an extension of [21] to nonsisothermal two-phase porous media flow problems. The presence of the temperature brings additional difficulties in obtaining a priori estimates and passage to the limit, and makes the proof essentially more involved. Indeed, the proof of the maximum principle for the temperature requires an appropriate combination of the three model equations. Moreover, as the system of equations is degenerate, obtaining estimates on the gradients is not straightforward. Therefore, we have introduced the nonisothermal global pressure and capillary term, because the latter present various regularity properties that physical pressures and saturation are lacking. Lastly, contrary to the isothermal case, there is a strong coupling induced by the energy equation and the pressure equation is no longer decoupled. This requires the use of some special test functions in order to control the energy of the system.

We have chosen the fully implicit CVFE discretization for many reasons. Indeed, the Euler implicit scheme is taken into account in order to avoid severe constraints on the time stepping. Furthermore, the possible low regularity of the saturation could make higher order schemes in time inefficient. With regards to the spatial discretization, the CVFE method has many advantages. First, it ensures the flux conservation at the interfaces between the control volumes. Second, it can deal well with anisotropies as only nodal degrees of freedom are required, contrary to many gradient schemes where auxiliary unknowns are employed. Lastly, it enables the coercivity of the gradient, see e.g. [18], which is an important key in the convergence analysis of the numerical scheme. Let us note that for other approaches like the MPFA (multi-point flux approximation) methods, the coercivity property holds under some conditions on both the mesh and the permeability tensor, see for instance [1, 18].

This paper is organized as follows: In Section 2, the mathematical formulation of the model is presented. Then we formulate the main assumptions on the data. The general finite volume discretization framework is brought up in Section 3. In Section 4, the discretization of the system provided by a fully coupled implicit approach using a finite volume (FV) scheme is presented. A number of auxiliary results, including the discrete maximum principle, energy estimates for the scheme, the existence of discrete solutions to the FV scheme and compactness, are demonstrated in Section 5. Finally, in Section 6, as discretization parameters go to zero, we pass to the limit in the discrete equations, the proposed scheme converges to a weak solution to the continuous two-phase flow model, which completes the convergence proof. Let us note that to reduce the paper's length, some technical details are removed, but a thorough analysis of the additional difficulties that temperature brings is provided. Then, in Section 7, we consider two test cases in 2D, for which numerical results are exhibited. The first test case is a simulation in a homogeneous reservoir evidencing the numerical convergence of the scheme. The second test case adapted from [28] relates to the injection of CO_2 into a 2D heterogeneous porous domain that is fully saturated with water. The paper is concluded in Section 8.

2. **Formulation of the problem**

We consider the nonisothermal flow of two immiscible incompressible fluids in a porous medium $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, capillary effect being taken into account. We set $\mathrm{Q} = \Omega \times]0, t_f[$ where $t_f > 0$ is a fixed time. For the presentation simplicity, we restrict the study to a horizontal field, i.e. the gravity effects are neglected. Now we will fix some notations and assumptions before giving the coupled system of PDEs modeling such flow. In the sequel, we denote by (w) the wetting phase, by (o) the nonwetting phase and by (s) the solid matrix.

(A.1) The porosity of the porous medium $\phi \in L^\infty(\Omega)$ and there exists two positive constants ϕ_1 and ϕ_2 such that: $\phi_1 \le \phi(x) \le \phi_2 < 1$ a.e. in Ω.

(A.2) The absolute permeability $\mathbb{K} \in (L^\infty(\Omega))^{d\times d}$ is a positive definite and symmetric tensor, and there exist constants such that $0 < K_1 < K_2$ and $K_1|\xi|^2 \le (\mathbb{K}(x)\xi)\cdot\xi \le K_2|\xi|^2$, $\forall\, \xi \in \mathbb{R}^d$ and a.e. in Ω. The effective thermal conductivity κ_T is assumed isotropic such that $\kappa_T = \kappa(x)\mathbb{I}$, where $\kappa \in L^\infty(\Omega)$ is a scalar function such that: $0 < K_1 \le \kappa(x) \le K_2$ a.e. in Ω.

(A.3) The densities of the wetting phase ρ_w, nonwetting phase ρ_o and the solid matrix ρ_s are positive constants.

(A.4) The capillary pressure function $P_c \in C^1([0,1];\mathbb{R}^+)$. Moreover, it is a nonincreasing function of the saturation, i.e., $P_c'(S) < 0$ in $[0,1]$ and $P_c(1) = 0$.

(A.5) The relative permeabilities k_{rw} and k_{ro} are in $C(\mathbb{R})$ and verify the following properties:
   (i) $0 \le k_{rw}, k_{ro} \le 1$ on $\mathbb{R}$;
   (ii) $k_{rw}(S) = 0$ for $S \le 0$ and $k_{ro}(S) = 0$ for $S \ge 1$; $k_{rw}(S) = 1$ for $S \ge 1$ and $k_{ro}(S) = 1$ for $S \le 0$;
   (iii) $k_{rw}(S) > 0$ and $k_{ro}(S) > 0$ for $S \in (0,1)$;
   (iv) there exists a positive constant k_0 such that $k_{rw}(S) + k_{ro}(S) \ge k_0$ for all $S \in \mathbb{R}$.

(A.6) The viscosities $\mu_w, \mu_o \in C^1(\mathbb{R})$ are functions of the temperature T, and there exists positive constants m_w, m_o, M_w and M_o such that for every $T \in \mathbb{R}$:

$$m_w \le \mu_w(T) \le M_w,\ |\mu_w'(T)| \le M_w; \quad m_o \le \mu_o(T) \le M_o,\ |\mu_o'(T)| \le M_o.$$

The mathematical model can be written as

$$\begin{aligned}
&0 \le S \le 1 && \text{in Q},\\
&P_o - P_w = P_c(S) && \text{in Q},\\
&\phi\frac{\partial S}{\partial t} - \operatorname{div}(\lambda_w \mathbb{K}\nabla P_w) = 0 && \text{in Q},\\
&-\phi\frac{\partial S}{\partial t} - \operatorname{div}(\lambda_o \mathbb{K}\nabla P_o) = 0 && \text{in Q},\\
&\frac{\partial(\psi T)}{\partial t} - \operatorname{div}(\lambda_w c_w T\mathbb{K}\nabla P_w) - \operatorname{div}(\lambda_o c_o T\mathbb{K}\nabla P_o) - \operatorname{div}(\kappa_T\nabla T) = 0 && \text{in Q},
\end{aligned}$$

where the unknowns are the phase pressures P_w and P_o, the wetting phase saturation S, and the temperature T. Furthermore, $\lambda_w = \frac{k_{rw}}{\mu_w}$, $\lambda_o = \frac{k_{ro}}{\mu_o}$ are respectively mobility of the phase w, o. We set $\lambda = \lambda_w + \lambda_o$, the total mobility and $\psi(\phi, S) = [c_w S + c_o(1-S)]\phi + c_s(1-\phi)$, where for $\alpha \in \{w, o, s\}$, $c_\alpha = \rho_\alpha C_\alpha$ with ρ_α, C_α are respectively the phase densities, the specific heat capacities for each phase.

As in [6, 13], the governing equations can be rewritten in a fractional flow formulation where the following new variable P called the nonisothermal global pressure is introduced:

$$P = P_o - \int_1^S \frac{\lambda_w}{\lambda}(s, T)P_c'(s)\,\mathrm{d}s.$$

Then, the mathematical model can be rewritten as (cf. [6, 13])

$$\begin{aligned}
&0 \le S \le 1 && \text{in Q},\\
&\phi\frac{\partial S}{\partial t} - \operatorname{div}(\lambda\eta_w\mathbb{K}\nabla P) - \operatorname{div}(\lambda\eta_w B_o\mathbb{K}\nabla T) - \operatorname{div}(\Lambda_0\mathbb{K}\nabla\beta(S)) = 0 && \text{in Q},\\
&-\phi\frac{\partial S}{\partial t} - \operatorname{div}(\lambda\eta_o\mathbb{K}\nabla P) - \operatorname{div}(\lambda\eta_o B_o\mathbb{K}\nabla T) + \operatorname{div}(\Lambda_0\mathbb{K}\nabla\beta(S)) = 0 && \text{in Q},\\
&\frac{\partial(\psi T)}{\partial t} - \operatorname{div}\{\lambda(\eta_w c_w + \eta_o c_o)T\mathbb{K}\nabla P\} + \operatorname{div}\{(c_o - c_w)\Lambda_0 T\mathbb{K}\nabla\beta(S)\}\\
&\qquad - \operatorname{div}\{\lambda(\eta_w c_w + \eta_o c_o)B_o T\mathbb{K}\nabla T\} - \operatorname{div}(\kappa_T\nabla T) = 0 && \text{in Q},
\end{aligned} \tag{2.1}$$

where the primary unknowns are P, the wetting phase saturation S and the temperature T. Furthermore, our model uses the following functions:

$$B_o(S,T) = -\int_1^S \frac{\partial}{\partial T}[\frac{\lambda_o}{\lambda}(s,T)]P_c'(s)\,\mathrm{d}s; \quad \Lambda_0(S,T) = \frac{M_o M_w}{m_o m_w}\frac{k_{ro}(S)m_w + k_{rw}(S)m_o}{k_{ro}(S)\mu_w(T) + k_{rw}(S)\mu_o(T)},$$

$$\eta_w = \frac{\lambda_w}{\lambda}; \eta_o = \frac{\lambda_o}{\lambda}; \quad \beta(S) = \int_1^S \gamma(s)\,\mathrm{d}s; \quad \gamma(S) = -\alpha(S)P_c'(S); \quad \alpha(S) = \frac{\frac{k_{rw}(S)}{M_w}\frac{k_{ro}(S)}{M_o}}{\frac{k_{rw}(S)}{m_w} + \frac{k_{ro}(S)}{m_o}},$$

where $\beta(S)$ is the Kirchhoff transformation [4]. Further assumptions on the physical data and nonlinearities are given in the following.

(A.7) β^{-1} (the inverse of the restriction of β to $[0,1]$) is a θ-Hölder function for some $\theta \in (0,1)$, on the interval $[0, \beta(1)]$, i.e.:

$$\exists\ C_\beta > 0\,, \quad \forall\, u_1, u_2 \in [0, \beta(1)] : |\beta^{-1}(u_1) - \beta^{-1}(u_2)| \le C_\beta|u_1 - u_2|^\theta.$$

(A.8) The initial data for the phases pressures are such that $P_o^0, P_w^0 \in L^2(\Omega)$ and $0 \le P_o^0 - P_w^0 \le P_c(0)$. Furthermore, the initial saturation $0 \le S^0 \le 1$ is defined by the capillary pressure law: $P_o^0 - P_w^0 = P_c(S^0)$. Moreover, the initial temperature $T^0 \in L^\infty(\Omega)$ and satisfies $T_{\min} \le T^0 \le T_{\max}$ a.e. in Ω, for some constants $T_{\min}$ and $T_{\max}$ such that $T_{\min} \le 0 \le T_{\max}$.

The system (2.1) has to be completed by appropriate boundary conditions. We assume that the boundary $\partial\Omega$ is comprised of two parts Γ_D and Γ_N such that $\Gamma_D \cap \Gamma_N = \emptyset$, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ and $|\Gamma_D| > 0$. The boundary conditions are as follows:

$$\begin{aligned} P = 0, \qquad [\beta(S)] = 0, \qquad T = 0 \quad \text{on } \Gamma_D \times (0, t_f),\\ \lambda_w(\mathbb{K}\nabla P_w).\vec{n} = 0, \quad \lambda_o(\mathbb{K}\nabla P_o).\vec{n} = 0, \quad (\kappa_T \nabla T).\vec{n} = 0 \quad \text{on } \Gamma_N \times (0, t_f). \end{aligned}$$

Remark 2.1.

(1) The assumptions (A.1)–(A.8) are classical and physically meaningful for two-phase flows in porous media. They are similar to the assumptions made in [6] that dealt with the existence of a weak solution of the studied problem.

(2) Assumptions (A.4) and (A.5) imply that the function $\gamma \in C([0,1];\mathbb{R}^+)$. Moreover, $\gamma(0) = \gamma(1) = 0$ and $\gamma > 0$ on $(0,1)$. Therefore, the restriction of β to $[0,1]$ is bijective.

(3) Assumptions (A.5) and (A.6) imply the existence of some positive constants λ_1 and λ_2 such that: $\lambda_1 \leq \lambda(S,T) \leq \lambda_2\,, \forall\, S, T \in \mathbb{R}$.

(4) In the mathematical model (2.1), we assumed that the densities are constant and that the porosity is only a function of the space variable. In addition, we consider a model with a single type of rock, i.e. the capillary pressure depends only on the saturation, and that it is not degenerate. Up to our knowledge, this is the first work and first step that deals with the convergence analysis for a CVFE scheme of nonisothermal flows in porous media. Due to the complexity of the equations and the coupling, the authors think that adding temperature dependency on the densities and porosity and the case of discontinuous and degenerate capillary pressures could render the analysis complicated and needs more paging to elaborate. Moreover, even the existence of weak solutions for the nonisothermal model under these assumptions is still an open problem. This could be treated in future contributions.

(5) Assumption (A.6) is in good accordance, for example, with the Reynolds model for shear viscosity (see, e.g., [30]). Namely, $\mu(T) = \mu_0 \exp(-bT)$, where μ_0 and b are constants.

3. CVFE mesh and notations

We assume that the porous medium Ω is a polygonal open subset of $\mathbb{R}^d$ (where d = 1, 2 or 3). We associate a primal mesh $\mathcal{T}$ to the domain Ω, which is a geometrically conforming simplicial triangulation in the finite elements sense, in other words, the intersection of 2 simplices is either a common face, edge, vertex or the empty set. We denote by $\mathcal{V}$ (resp. $\mathcal{V}_\tau$) the set of all nodes (vertices) of $\mathcal{T}$ (resp. $\tau \in \mathcal{T}$). For a given simplex $\tau \in \mathcal{T}$, x_τ denotes the barycenter of τ, $h_\tau = \operatorname{diam}(\tau)$ the diameter and $|\tau|$ its Lebesgue measure. Moreover, let ϱ_τ be the diameter of the inscribed circle of the simplex τ. We denote by h and $\theta_{\mathcal{T}}$, the size and the regularity of the partition $\mathcal{T}$, respectively. They are defined as follows:

$$h := \max_{\tau\in\mathcal{T}}(h_\tau) \quad \text{and} \quad \theta_{\mathcal{T}} := \max_{\tau\in\mathcal{T}} \frac{h_\tau}{\varrho_\tau}.$$

For a given node $L \in \mathcal{V}$, we denote by τ_L the set of all primal mesh simplices with the common vertex L. Furthermore, we construct a barycentric dual mesh in the following manner. For every node $L \in \mathcal{V}$, we associate a unique control volume denoted by ω_L, of the dual mesh. Each dual control volume is made by connecting the barycenter of every $\tau \in \tau_L$ to the midpoint of edges of τ which have L as a vertex. For 2 vertices $L, M \in \mathcal{V}_\tau$, we denote by σ^τ_{LM} the dual interface located inside τ and intersects the segment $[LM]$. We denote by $|\sigma^\tau_{LM}|$ the $d-1$ Lebesgue measure of the interface

σ_{LM}^{τ} and by $\vec{n}_{\sigma_{LM}^{\tau}}$ the unit normal to σ_{LM}^{τ} pointing from L to M. We also denote $\mathcal{E}_{\tau}$ the set of dual interfaces that are inside τ. Moreover, for $L \in \mathcal{V}$, $|\omega_L|$ denotes the d-Lebesgue measure of ω_L.

Let us state the main assumptions made about the mesh.

(A.9) We assume the regularity of the primal mesh, i.e. there exists some constant C_0 such that for every sequence of discretizations $\{\mathcal{T}_m\}_{m\in\mathbb{N}}$ of the initial mesh, there holds: $\theta_{\mathcal{T}_m} \leq C_0$.

(A.10) We assume that there exists some constant C_1 such that for every $\tau \in \mathcal{T}_m$ and every $L, M \in \mathcal{V}_{\tau}$: $\mathrm{diam}(\omega_L) + \mathrm{diam}(\omega_M) \leq C_1 \mathrm{d}(x_L, x_M)$, where $\mathrm{diam}(\omega_L)$ is the diameter of ω_L and $\mathrm{d}(x_L, x_M)$ is the Euclidean distance between L and M.

(A.11) We assume in $2D$ that the triangulation $\mathcal{T}_m$ is weakly acute, i.e. no triangle has an angle greater than $\pi/2$ and in 3D the faces of the tetra elements fulfill the angle condition. In other words, the dihedral angles must be nonobtuse in order that the discrete maximum principle holds for the solution of the temperature. Example of meshes satisfying this condition are described in [9, 26].

In regards to the time interval $[0,T]$, we consider the following discretization $(t^n)_{n\in[\![0,l]\!]}$: $0 = t^0 < t^1 < \cdots < t^{l-1} < t^l = t_f$. We set $\delta t = t^{n+1} - t^n$ for $n \in [\![0, l-1]\!]$ and define $\delta t := \max_{n\in[\![0,l-1]\!]}(\delta t^n)$. For the sake of simplicity, we consider a constant time step. Our approach is easily extendable to variable time stepping by assuming that $\alpha\delta t < \min_{n\in[\![0,l-1]\!]}(\delta t^n)$ for some $\alpha \in (0,1)$. In practice, the time step remains between $\delta t_{\min}$ and $\delta t_{\max}$.

For the numerical approximation of our model (2.1), we will use an implicit Euler scheme in time and a vertex-centered finite volume scheme in space. An upwind method is used to treat the convection terms and $\mathbb{P}_1$ finite elements are used for the approximation of the gradients.

Now, let us define the approximation spaces for the discrete unknowns. Additionally, we will describe the construction of the discrete functions. In this sense, let us denote by X_h the space of piecewise linear functions on the primal mesh and by W_h the space of piecewise constant functions on the dual mesh. Therefore, X_h and W_h are finite-dimensional spaces. Moreover, there holds:

$$X_h = \{\varphi \in C^0(\overline{\Omega}) : \varphi_{|\tau} \in \mathbb{P}_1, \forall\, \tau \in \mathcal{T}\} \subset H^1(\Omega).$$

Let us define the space

$$X_h^0 = \{\varphi \in X_h : \varphi(x_L) = 0, \forall\, L \in \mathcal{V} \text{ such that } x_L \in \Gamma_D\}.$$

Under the assumption that Γ_D is polygonal and that its limiting vertices belong to $\mathcal{V}$, the following inclusion holds $X_h^0 \subset H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \text{ such that } v_{|\Gamma_D} = 0\}$. Next, we denote by $(\varphi_L)_{L\in\mathcal{V}}$ the global basis functions which are associated to each node in the finite elements sense. For $L, M \in \mathcal{V}$, there holds

$$\varphi_L(x_M) = \delta_{LM}, \quad \text{(Kronecker delta)}.$$

For $\tau \in \mathcal{T}$, one has: $\sum_{L\in\mathcal{V}_\tau} \varphi_{L|_\tau} = \mathbb{1}_\tau$. Furthermore, the following formulas hold:

$$\sum_{L\in\mathcal{V}} \varphi_L = 1, \quad \sum_{L\in\mathcal{V}} \nabla\varphi_L = 0 \quad \text{and} \quad \nabla\varphi_{L|_\tau} = -\frac{|\sigma_L^\tau|}{2|\tau|}\vec{n}_{\sigma_L^\tau},$$

where σ_L^τ is the face of triangle τ facing vertex L and $\vec{n}_{\sigma_L^\tau}$ the unit outer normal associated to the boundary of τ. In the sequel, we define:

$$S_L^0 = \frac{1}{|\omega_L|}\int_{\omega_L} S(x,0)\mathrm{d}x, \quad P_L^0 = \frac{1}{|\omega_L|}\int_{\omega_L} P(x,0)\mathrm{d}x, \quad T_L^0 = \frac{1}{|\omega_L|}\int_{\omega_L} T(x,0)\mathrm{d}x.$$

The unknowns are $(S_L^n, P_L^n, T_L^n)_{L\in\mathcal{V}}$ for $n \in [\![1, l]\!]$, where

$$S_L^n \simeq \frac{1}{|\omega_L|}\int_{\omega_L} S(x,t^n)\mathrm{d}x, \quad P_L^n \simeq \frac{1}{|\omega_L|}\int_{\omega_L} P(x,t^n)\mathrm{d}x, \quad T_L^n \simeq \frac{1}{|\omega_L|}\int_{\omega_L} T(x,t^n)\mathrm{d}x.$$

Finally, let F be a function of (S, P, T). We denote by $\big[\widetilde{F}(S;P;T)\big]_{h,\delta t}(x,t)$ the finite volume reconstruction which is defined almost everywhere in Q, and by $[F(S;P;T)]_{h,\delta t}(x,t)$ the finite element reconstruction, i.e.

$$[\widetilde{F}(S;P;T)]_{h,\delta t}(x,t) := \sum_{L\in\mathcal{V}} \mathbb{1}_{\omega_L}(x)F(S_L^0;P_L^0;T_L^0)\mathbb{1}_{\{0\}}(t) + \sum_{L\in\mathcal{V}} \mathbb{1}_{\omega_L}(x)\sum_{n=0}^{l-1} F(S_L^{n+1};P_L^{n+1};T_L^{n+1})\mathbb{1}_{(t^n,t^{n+1}]}(t),$$

$$[F(S;P;T)]_{h,\delta t}(x,t) := \sum_{L\in\mathcal{V}} F(S_L^0;P_L^0;T_L^0)\varphi_L(x)\mathbb{1}_{\{0\}}(t) + \sum_{L\in\mathcal{V}}\sum_{n=0}^{l-1} F(S_L^{n+1};P_L^{n+1};T_L^{n+1})\varphi_L(x)\mathbb{1}_{(t^n,t^{n+1}]}(t).$$

4. **Finite volume discretization**

For the discretization of the coupled system (2.1), we first integrate the three equations of (2.1) over ω_L (for $L \in \mathcal{V}$) and then in time over the interval $[t^n, t^{n+1})$ (for $n \in [\![0, l-1]\!]$). Afterwards, we apply the Green formula and approximate properly the fluxes at each interface $\partial\omega_L$.

After integrating the first equation of (2.1) in space over ω_L and in time over the interval $[t^n, t^{n+1})$, then by Green's formula, we get

$$A_1 + A_2 + A_3 + A_4 = 0,$$

where

$$A_1 = |\omega_L|\phi_L(S_L^{n+1} - S_L^n), \qquad A_2 = -\int_{t^n}^{t^{n+1}} \delta t \int_{\partial\omega_L} \lambda\eta_w(\mathbb{K}\nabla P)\cdot\vec{n}_{\omega_L}\mathrm{d}\Gamma,$$

$$A_3 = -\int_{t^n}^{t^{n+1}} \delta t \int_{\partial\omega_L} \lambda\eta_w(B_o^+ - B_o^-)(\mathbb{K}\nabla T)\cdot\vec{n}_{\omega_L}\mathrm{d}\Gamma, \qquad A_4 = -\int_{t^n}^{t^{n+1}} \delta t \int_{\partial\omega_L} \Lambda_0(\mathbb{K}\nabla\beta(S))\cdot\vec{n}_{\omega_L}\mathrm{d}\Gamma.$$

We recall the following notation: for $c \in \mathbb{R}$, $c^+ := \max(c, 0)$ and $c^- := \max(-c, 0)$. Moreover, we have denoted $\phi_L = \frac{1}{|\omega_L|}\int_{\omega_L}\phi(x)\mathrm{d}x = \frac{1}{|\omega_L|}\sum_{\tau\in\tau_L}\int_{\tau\cap\omega_L}\phi(x)\mathrm{d}x$. In the sequel, we discretize each of the terms A_2, A_3 and A_4. As for the term A_2, we use the following approximation:

$$A_2 \simeq -\delta t \sum_{\tau\in\tau_L}\sum_{M\in\mathcal{V}_\tau\setminus\{L\}} \lambda_\tau^{n+1} G_w((S;T)_L^{n+1};(S;T)_M^{n+1};\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P,$$

where

$$\lambda_\tau^{n+1} := \frac{1}{\#\mathcal{V}_\tau}\sum_{M\in\mathcal{V}_\tau}\lambda(S_M^{n+1},T_M^{n+1}), \mathbb{K}_{LM}^\tau := -\int_\tau(\mathbb{K}\nabla\varphi_L)\cdot\nabla\varphi_M\mathrm{d}x, \delta_{LM}^{n+1}P = P_M^{n+1} - P_L^{n+1},$$

and

$$G_w(a;b;c) := \begin{cases}\eta_w(b) & \text{if } c \geq 0,\\ \eta_w(a) & \text{otherwise.}\end{cases}$$

Similarly to A_2, we use the following approximation for A_3:

$$A_3 \simeq \begin{bmatrix} -\delta t \displaystyle\sum_{\tau\in\tau_L}\sum_{M\in\mathcal{V}_\tau\setminus\{L\}} \lambda_\tau^{n+1} H_w^1((S;T)_L^{n+1};(S;T)_M^{n+1};\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T \\ +\delta t \displaystyle\sum_{\tau\in\tau_L}\sum_{M\in\mathcal{V}_\tau\setminus\{L\}} \lambda_\tau^{n+1} H_w^2((S;T)_L^{n+1};(S;T)_M^{n+1};-\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T \end{bmatrix},$$

where

$$H_w^1(a;b;c) := \begin{cases} [\eta_w B_o^+](b) & \text{if } c \geq 0, \\ [\eta_w B_o^+](a) & \text{otherwise}, \end{cases} \qquad H_w^2(a;b;c) := \begin{cases} [\eta_w B_o^-](b) & \text{if } c \geq 0, \\ [\eta_w B_o^-](a) & \text{otherwise}. \end{cases}$$

Before discretizing the term A_4, recall that there exists some constants C_2, $\Lambda_{0,\min}$ and $\Lambda_{0,\max}$ such that for every $S, T \in \mathbb{R}$:

$$|B_o(S,T)| \leq C_2 \quad \text{and} \quad 0 < \Lambda_{0,\min} \leq \Lambda_0(S;T) \leq \Lambda_{0,\max} < +\infty. \tag{4.1}$$

By assuming that S is regular enough, we deduce that: $\nabla\beta(S) = -\alpha(S)P_c'(S)\nabla S$.

Hence by defining: $\gamma(S) := -\alpha(S)P_c'(S)$, we get:

$$\Lambda_0(\mathbb{K}\nabla\beta(S)) = \Lambda_0(S;T)\gamma(S)(\mathbb{K}\nabla S).$$

Lastly, regarding the term A_4, we use the following approximation:

$$A_4 \simeq -\delta t \sum_{\tau\in\tau_L} \sum_{M\in\mathcal{V}_\tau\setminus\{L\}} [\Lambda_0]_\tau^{n+1} \gamma_{LM}^{n+1} \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} S,$$

where

$$[\Lambda_0]_\tau^{n+1} := \frac{1}{\#\mathcal{V}_\tau} \sum_{M\in\mathcal{V}_\tau} \Lambda_0(S_M^{n+1}; T_M^{n+1}) \quad \text{and} \quad \gamma_{LM}^{n+1} := \begin{cases} \max_{S\in I_{LM}^{n+1}} \gamma(S) & \text{if } \mathbb{K}_{LM}^\tau \geq 0, \\ \min_{S\in I_{LM}^{n+1}} \gamma(S) & \text{otherwise}, \end{cases}$$

where $I_{LM}^{n+1} = [S_L^{n+1}, S_M^{n+1}]$.

Finally, the discretization of the first equation of system (2.1) writes as follows:

$$\frac{|\omega_L|}{\delta t}\phi_L(S_L^{n+1} - S_L^n) - \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} \eta_w(;^P) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P - \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1} \gamma_{LM}^{n+1} \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} S$$
$$- \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} ([\eta_w B_o^+](;^T) - [\eta_w B_o^-](;^{-T})) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T = 0.$$

For the sake of legibility, we introduced the following notation: for some function f of variables S and T, some node $L \in \mathcal{V}$, simplex $\tau \in \tau_L$, node $M \in \mathcal{V}_\tau \setminus \{L\}$ and $H = P$, T or "$-T$", we denote

$$f(;^H) = G_f((S;T)_L^{n+1}; (S;T)_M^{n+1}; \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} H), \text{and} G_f(a;b;c) := \begin{cases} f(b) & \text{if } c \geq 0, \\ f(a) & \text{otherwise}, \end{cases}$$

where $\delta_{LM}^{n+1}(-T) := -T_M^{n+1} + T_L^{n+1} = -\delta_{LM}^{n+1} T$.

In the same way, we discretize the other two equations. Finally, the numerical scheme for the coupled system (2.1) writes as follows:

$$\frac{|\omega_L|}{\delta t}\phi_L(S_L^{n+1} - S_L^n) - \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} \eta_w(;^P) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P - \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1} \gamma_{LM}^{n+1} \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} S$$
$$- \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} ([\eta_w B_o^+](;^T) - [\eta_w B_o^-](;^{-T})) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T = 0,$$

$$-\frac{|\omega_L|}{\delta t}\phi_L(S_L^{n+1}-S_L^n) - \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}\eta_o(;^P)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P + \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S$$
$$- \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}([\eta_o B_o^+](;^T) - [\eta_o B_o^-](;^{-T}))\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T = 0, \quad (4.2)$$

$$\frac{|\omega_L|}{\delta t}[\psi(\phi_L;S_L^{n+1})T_L^{n+1} - \psi(\phi_L;S_L^n)T_L^n] - \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)T](;^P)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P$$
$$- \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^+T](;^T)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T + \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^-T](;^{-T})\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T$$
$$- \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (c_w[T](;^S) - c_o[T](;^{-S}))[\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S - \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (\kappa_T)_{LM}^\tau\delta_{LM}^{n+1}T = 0,$$

where $(\kappa_T)_{LM}^\tau := -\int_\tau(\kappa_T\nabla\varphi_L)\cdot\nabla\varphi_M\mathrm{d}x$.

5. Stability properties and existence of the numerical scheme

5.1. Maximum principle for the saturation

The techniques used for the proof of the maximum principle for the saturation are similar with the ones in the isothermal case as done for instance in [21], and are therefore skipped here.

Proposition 5.1. *Let* $(S_L^{n+1}, P_L^{n+1}, T_L^{n+1})_{L\in\mathcal{V},n\in[\![0,l-1]\!]}$ *be a solution to the numerical scheme* (4.2)*. If* $(S_L^0)_{L\in\mathcal{V}}$ *belongs to* $[0,1]$*, then* $(\widetilde{S}_{h,\delta t})$ *remains in* $[0,1]$.

5.2. Maximum principle for the temperature

Now, we turn to the proof of the maximum principle for the temperature.

Proposition 5.2. *Under the assumptions* (A.1) *and* (A.11)*, let* $(S_L^{n+1}, P_L^{n+1}, T_L^{n+1})_{L\in\mathcal{V},n\in[\![0,l-1]\!]}$ *be a solution to the numerical scheme* (4.2)*. Then* $(\widetilde{T}_{h,\delta t})$ *remains in* $[T_{\min}, T_{\max}]$.

Proof. We proceed by induction on n. The property is trivial for $n = 0$. Now, assume that the sequence $(T_L^n)_{L\in\mathcal{V}} \subset [T_{\min}, T_{\max}]$ for all $n \in [\![0, l-2]\!]$ and let us prove that $(T_L^{n+1})_{L\in\mathcal{V}} \subset [T_{\min}, T_{\max}]$. For some fixed $L \in \mathcal{V}$ we multiply the 1st equation of (4.2) by $[-c_wT_L^{n+1}]$ and the 2nd equation by $[-c_oT_L^{n+1}]$. Adding up the two equations gives:

$$-|\omega_L|[\psi(\phi_L;S_L^{n+1}) - \psi(\phi_L;S_L^n)]T_L^{n+1} + \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[c_w\eta_w + c_o\eta_o](;^P)T_L^{n+1}\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P$$
$$+\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (c_w - c_o)T_L^{n+1}[\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S + \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^+](;^T)T_L^{n+1}\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T$$
$$+ \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^-](;^{-T})T_L^{n+1}(-\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T) = 0. \quad (5.1)$$

Indeed, one has:

$$\begin{aligned}\psi(\phi_L;S_L^{n+1})-\psi(\phi_L;S_L^{n}) &= [c_w S_L^{n+1}+c_o(1-S_L^{n+1})]\phi_L + c_s(1-\phi_L)\\ &\quad -\{[c_w S_L^{n}+c_o(1-S_L^{n})]\phi_L + c_s(1-\phi_L)\}\\ &= [c_w(S_L^{n+1}-S_L^{n})-c_o(S_L^{n+1}-S_L^{n})]\phi_L\\ &= \phi_L(c_w-c_o)(S_L^{n+1}-S_L^{n}).\end{aligned}$$

Adding up equation (5.1) and the 3rd equation of (4.2), one gets:

$$A_L^{n+1}=0, \tag{5.2}$$

where

$$\begin{aligned}A_L^{n+1} &:= |\omega_L|\psi(\phi_L;S_L^{n})(T_L^{n+1}-T_L^{n}) - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} [c_w\eta_w+c_o\eta_o](;^{P})([T](;^{P})-T_L^{n+1})\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}P\\ &-\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} c_w([T](;^{S})-T_L^{n+1})[\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}S\\ &+\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} c_o([T](;^{-S})-T_L^{n+1})[\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}S\\ &-\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^{+}](;^{T})([T](;^{T})-T_L^{n+1})\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}T\\ &-\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^{-}](;^{-T})([T](;^{-T})-T_L^{n+1})(-\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}T) - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (\kappa_T)_{LM}^{\tau}\delta_{LM}^{n+1}T.\end{aligned}$$

Now, note that for $\tau\in\tau_L$, $M\in\mathcal{V}_\tau\setminus\{L\}$ and $H=P$, T, "$-T$", S or "$-S$", one writes

$$[T](;^{H})\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H = T_M^{n+1}(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{+} - T_L^{n+1}(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{-},$$

Therefore, one infers

$$\begin{aligned}([T](;^{H})-T_L^{n+1})\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H &= [T_M^{n+1}(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{+} - T_L^{n+1}(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{-}]\\ &\quad - T_L^{n+1}[(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{+} - (\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{-}]\\ &= (T_M^{n+1}-T_L^{n+1})(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{+}\\ &= \delta_{LM}^{n+1}T(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}H)^{+}.\end{aligned}$$

This allows us to reformulate A_L^{n+1} as follows

$$\begin{aligned}A_L^{n+1} &= |\omega_L|\psi(\phi_L;S_L^{n})(T_L^{n+1}-T_L^{n}) - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[c_w\eta_w+c_o\eta_o](;^{P})\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}P)^{+}\\ &\quad -\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\delta_{LM}^{n+1}T[c_w(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}S)^{+} + c_o(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}S)^{-}]\\ &\quad -\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^{+}](;^{T})\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}T)^{+}\\ &\quad -\delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^{-}](;^{-T})\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^{\tau}\delta_{LM}^{n+1}T)^{-} - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (\kappa_T)_{LM}^{\tau}\delta_{LM}^{n+1}T.\end{aligned}$$

Now, let $L \in \mathcal{V}$ be such that $T_L^{n+1} = \min\{T_M^{n+1}\}_{M\in\mathcal{V}}$. For every $\tau \in \tau_L$ and $M \in \mathcal{V}_\tau \setminus \{L\}$, one has $-\delta_{LM}^{n+1}T \leq 0$. As a consequence

$$
\begin{aligned}
&- \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[c_w\eta_w + c_o\eta_o](;^P)\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P)^+ \\
&\qquad - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1}\gamma_{LM}^{n+1}\delta_{LM}^{n+1}T[c_w(\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S)^+ + c_o(\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S)^-] \\
&\qquad - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^+](;^T)\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T)^+ \\
&\qquad - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}[(c_w\eta_w + c_o\eta_o)B_o^-](;^{-T})\delta_{LM}^{n+1}T(\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T)^- \\
&\qquad\qquad\qquad - \delta t \sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}} (\kappa_T)_{LM}^\tau\delta_{LM}^{n+1}T \leq 0.
\end{aligned}
$$

Note that by virtue of assumption (A.11), $(\kappa_T)_{LM}^\tau \geq 0$. Moreover, all the involved quantities are nonnegative. Taking into account (5.2), there holds

$$|\omega_L|\psi(\phi_L; S_L^n)(T_L^{n+1} - T_L^n) \geq 0.$$

Thus, noting that $\psi(\phi_L; S_L^n) > 0$, one finally obtains $T_L^{n+1} \geq T_L^n \geq T_{\min}$. As a result, $T_M^{n+1} \geq T_{\min}$, $\forall\, M \in \mathcal{V}$. Similarly, one can establish the upper bound on the T_M^{n+1}, which concludes the proof. ■

5.3. **A priori estimates for the gradients**

The following Lemmas were both stated in [21]. The first one is required to tackle the difficulty induced by negative transmissibility coefficients. The second one is nothing more than the discrete integration by parts.

Lemma 5.3. *Under the assumption* (A.9)*, there exists some constant C_K, depending only on $\mathbb{K}$ and C_0, such that:*

$$\forall\, u_h = \sum_{L\in\mathcal{V}} u_L\varphi_L \in X_h : \sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} |\mathbb{K}_{LM}^\tau|(u_M - u_L)^2 \leq C_K \int_\Omega (\mathbb{K}\nabla u_h)\cdot\nabla u_h \mathrm{d}x.$$

Lemma 5.4. *For every $u_h, v_h \in X_h$, there holds:*

$$\int_\Omega (\mathbb{K}\nabla u_h)\cdot\nabla v_h\, \mathrm{d}x = \sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(u_M - u_L)(v_M - v_L).$$

Proposition 5.5. *Under the assumptions* (A.1)–(A.11)*, let $(S_L^{n+1}, P_L^{n+1}, T_L^{n+1})_{L\in\mathcal{V}, n\in[\![0,l-1]\!]}$ be a solution to the numerical scheme* (4.2)*. Then, one has:*

$$
\begin{aligned}
A_T &:= \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} (\kappa_T)_{LM}^\tau(\delta_{LM}^{n+1}T)^2 \\
&\leq \frac{1}{2}\sum_{L\in\mathcal{V}} |\omega_L|[\psi(\phi_L; S_L^0)(T_L^0)^2 - \psi(\phi_L; S_L^l)(T_L^l)^2].
\end{aligned}
\tag{5.3}
$$

Moreover, there exist some constants C_T, C_P and C_β, depending only on the constants $|\Omega|$, ρ_w, ρ_o, ρ_s, c_w, c_o, c_s, K_1, K_2, λ_1, λ_2, $\|T^0\|_{L^2(\Omega)}$, C_K, C_2 and $\Lambda_{0,\min}$, such that:

$$\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} (\kappa_T)_{LM}^\tau (\delta_{LM}^{n+1} T)^2 \leq C_T, \tag{5.4}$$

$$\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau (\delta_{LM}^{n+1} P)^2 \leq C_P, \tag{5.5}$$

$$\sum_{L\in\mathcal{V}} |\omega_L| \phi_L [\varphi(S_L^l) - \varphi(S_L^0)] + \frac{\Lambda_{0,\min}}{2} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau (\delta_{LM}^{n+1} \beta(S))^2 \leq C_\beta, \tag{5.6}$$

where $\varphi'(s) = \beta(s)$.

Proof. The proof is conducted in several steps.

• **Temperature estimation.** We start by multiplying the discrete energy equation of (4.2) by T_L^{n+1}, then we sum over $L \in \mathcal{V}$ and $n \in [\![0, l-1]\!]$. This gives

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0, \tag{5.7}$$

where we rearranged each summation by simplices by the dual interfaces to obtain

$$A_1 = \sum_{L\in\mathcal{V}} |\omega_L| \sum_{n=0}^{l-1} [\psi(\phi_L; S_L^{n+1})(T_L^{n+1})^2 - \psi(\phi_L; S_L^n) T_L^n T_L^{n+1}],$$

$$A_2 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} [c_w\eta_w + c_o\eta_o](;^P)[T](;^P) \delta_{LM}^{n+1} T \cdot \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P,$$

$$A_3 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} [\Lambda_0]_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} (c_w[T](;^S) - c_o[T](;^{-S})) \gamma_{LM}^{n+1} \delta_{LM}^{n+1} T \cdot \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} S,$$

$$A_4 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} [(c_w\eta_w + c_o\eta_o) B_o^+](;^T)[T](;^T) \delta_{LM}^{n+1} T \cdot \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T,$$

$$A_5 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} [(c_w\eta_w + c_o\eta_o) B_o^-](;^{-T})[T](;^{-T}) \delta_{LM}^{n+1} T \cdot (-\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T),$$

$$A_6 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} (\kappa_T)_{LM}^\tau (\delta_{LM}^{n+1} T)^2.$$

Next, we multiply the 1st equation of (4.2) by $[-\frac{1}{2} c_w (T_L^{n+1})^2]$, we sum over $L \in \mathcal{V}$ and $n \in [\![0, l-1]\!]$. One has

$$B_1 + B_2 + B_3 + B_4 + B_5 = 0, \tag{5.8}$$

where

$$B_1 = -\frac{1}{2} \sum_{L\in\mathcal{V}} |\omega_L| \sum_{n=0}^{l-1} \phi_L c_w (S_L^{n+1} - S_L^n)(T_L^{n+1})^2,$$

$$B_2 = -\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_w \eta_w(;^P)[T]_{LM}^{n+1} \delta_{LM}^{n+1} T \cdot \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P,$$

$$B_3 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\lambda_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_w[\eta_w B_o^+](;^T)[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T,$$

$$B_4 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\lambda_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_w[\eta_w B_o^-](;^{-T})[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot(-\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T),$$

$$B_5 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}[\Lambda_0]_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_w\gamma_{LM}^{n+1}[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S.$$

We used the following notation

$$[T]_{LM}^{n+1} := \frac{T_L^{n+1}+T_M^{n+1}}{2}.$$

At last, we multiply the 2nd equation of (4.2) by $[-\frac{1}{2}c_o(T_L^{n+1})^2]$, we sum over $L\in\mathcal{V}$ and $n\in[\![0,l-1]\!]$. The resulting equation reads

$$G_1+G_2+G_3+G_4+G_5=0, \tag{5.9}$$

where

$$G_1 = -\frac{1}{2}\sum_{L\in\mathcal{V}}|\omega_L|\sum_{n=0}^{l-1}\phi_L c_o[(1-S_L^{n+1})-(1-S_L^n)](T_L^{n+1})^2,$$

$$G_2 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\lambda_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_o\eta_o(;^P)[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P,$$

$$G_3 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\lambda_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_o[\eta_o B_o^+](;^T)[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T,$$

$$G_4 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\lambda_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_o[\eta_o B_o^-](;^{-T})[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot(-\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T),$$

$$G_5 = -\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}[\Lambda_0]_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} c_o\gamma_{LM}^{n+1}[T]_{LM}^{n+1}\delta_{LM}^{n+1}T\cdot(-\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}S).$$

Finally, adding up the relationships (5.7), (5.8) and (5.9), one gets

$$H_1+H_2+H_3+H_4+H_5+H_6=0, \tag{5.10}$$

where

$$H_1 = A_1+B_1+G_1;\quad H_2 = A_2+B_2+G_2;\quad H_3 = A_3+B_5+G_5,$$
$$H_4 = A_4+B_3+G_3;\quad H_5 = A_5+B_4+G_4;\quad H_6 = A_6.$$

Let us compute H_1. Observe that

$$B_1+G_1 = -\frac{1}{2}\sum_{L\in\mathcal{V}}|\omega_L|\sum_{n=0}^{l-1}[\psi(\phi_L;S_L^{n+1})-\psi(\phi_L;S_L^n)](T_L^{n+1})^2.$$

Therefore

$$H_1 = \frac{1}{2}\sum_{L\in\mathcal{V}}|\omega_L|[\psi(\phi_L;S_L^l)(T_L^l)^2-\psi(\phi_L;S_L^0)(T_L^0)^2]+\frac{1}{2}\sum_{L\in\mathcal{V}}|\omega_L|\sum_{n=0}^{l-1}\psi(\phi_L;S_L^n)(T_L^{n+1}-T_L^n)^2.$$

Now, the fact that $\psi(\phi; S) \geq 0$ implies that:

$$H_1 \geq \frac{1}{2} \sum_{L\in\mathcal{V}} |\omega_L| [\psi(\phi_L; S_L^l)(T_L^l)^2 - \psi(\phi_L; S_L^0)(T_L^0)^2]. \tag{5.11}$$

Before computing the terms H_2, H_3, H_4 and H_5, note that for $\tau \in \tau_L$, $M \in \mathcal{V}_\tau \setminus \{L\}$ and $H = P$, T, "$-T$", S or "$-S$", the following identity is satisfied

$$\begin{aligned}([T](;^H) - [T]_{LM}^{n+1})\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} H &= (T_M^{n+1} - [T]_{LM}^{n+1})(\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} H)^+ - (T_L^{n+1} - [T]_{LM}^{n+1})(\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} H)^- \\ &= \frac{1}{2}\delta_{LM}^{n+1} T |\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} H|.\end{aligned}$$

As a result

$$H_2 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} [c_w\eta_w + c_o\eta_o](;^P)(\delta_{LM}^{n+1} T)^2 \cdot |\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P|,$$

$$H_3 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} [\Lambda_0]_\tau^{n+1} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} (c_w + c_o)\gamma_{LM}^{n+1} (\delta_{LM}^{n+1} T)^2 \cdot |\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} S|,$$

$$H_4 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} [(c_w\eta_w + c_o\eta_o)B_o^+](;^T)(\delta_{LM}^{n+1} T)^2 \cdot |\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T|,$$

$$H_5 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} [(c_w\eta_w + c_o\eta_o)B_o^-](;^{-T})(\delta_{LM}^{n+1} T)^2 \cdot |\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T|.$$

It is clear that the quantities H_2, H_3, H_4 and H_5 are positive. Therefore, from (5.10) and (5.11), we deduce

$$A_6 + \frac{1}{2} \sum_{L\in\mathcal{V}} |\omega_L| [\psi(\phi_L; S_L^l)(T_L^l)^2 - \psi(\phi_L; S_L^0)(T_L^0)^2] \leq 0.$$

The latter inequality entails the first estimation

$$A_6 = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} (\kappa_T)_{LM}^\tau (\delta_{LM}^{n+1} T)^2 \leq \frac{1}{2} \sum_{L\in\mathcal{V}} |\omega_L| [\psi(\phi_L; S_L^0)(T_L^0)^2 - \psi(\phi_L; S_L^l)(T_L^l)^2].$$

Now, noting that $0 \leq S_L^0 \leq 1$ and $0 \leq \phi \leq 1$, one gets

$$\psi(\phi_L; S_L^0) = [c_w S_L^0 + c_o(1 - S_L^0)]\phi_L + c_s(1 - \phi_L) \leq c_w + c_o + c_s := C_3.$$

The discrete integration by part together with Jensen's inequality yield

$$\int_{\mathrm{Q}} (\kappa_T \nabla T_{h,\delta t}) \cdot \nabla T_{h,\delta t} \,\mathrm{d}x\,\mathrm{d}t \leq \frac{C_3 \|T^0\|_{L^2(\Omega)}}{2}.$$

Thus

$$\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} (\kappa_T)_{LM}^\tau (\delta_{LM}^{n+1} T)^2 \leq C_4, \tag{5.12}$$

for some constant $C_4 > 0$ independent of the discretization steps.

- **Global pressure estimation.** We first sum the 1st and 2nd equation of (4.2). Then

$$-\delta t \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P - \delta t \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}([B_o^+](;^T) - [B_o^-](;^{-T}))\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T = 0.$$

Next, we multiply this equality by (P_L^{n+1}) then we sum over $L \in \mathcal{V}$ and $n \in [\![0, l-1]\!]$ to get

$$D_1 + D_2 = 0, \tag{5.13}$$

where

$$D_1 = -\sum_{n=0}^{l-1} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1}\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1}P\, P_L^{n+1} = \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2,$$

$$D_2 = \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} ([B_o^+](;^T) - [B_o^-](;^{-T}))\mathbb{K}_{LM}^\tau \delta_{LM}^{n+1}P\delta_{LM}^{n+1}T.$$

Now, recall that $0<\lambda_1\leq\lambda$ and that for $\tau\in\mathcal{T}$, thanks to Lemma 5.4 one has $\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}\mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2\geq 0$. As a consequence, $\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}\mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2\geq 0$ for every $\tau\in\mathcal{T}$ and thus

$$D_1 \geq \lambda_1 \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2. \tag{5.14}$$

Notice that $|B_o|\leq C_2$ and $0<\lambda\leq\lambda_2$. By virtue of the triangle inequality:

$$|D_2| \leq 2\,C_2\lambda_2 \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} |\mathbb{K}_{LM}^\tau||\delta_{LM}^{n+1}P||\delta_{LM}^{n+1}T|.$$

Therefore, for $\varepsilon>0$, one has in light of Young's inequality:

$$|D_2| \leq 2\varepsilon C_2\lambda_2 \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} |\mathbb{K}_{LM}^\tau|(\delta_{LM}^{n+1}P)^2 + \frac{C_2\lambda_2}{2\varepsilon}\sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} |\mathbb{K}_{LM}^\tau|(\delta_{LM}^{n+1}T)^2.$$

According to Lemma 5.3, one can set $\varepsilon = \frac{\lambda_1}{4\,C_2\lambda_2 C_K}$ and deduce from (5.4) that

$$|D_2| \leq \frac{\lambda_1}{2}\sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2 + \frac{2\,(C_2\lambda_2 C_K)^2}{\lambda_1}K_2C_4'.$$

Here $C_4' = C_4K_2/K_1$. Hence, it follows from (5.14) that

$$\sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}P)^2 \leq 4\left(\frac{C_2\lambda_2C_K}{\lambda_1}\right)^2 K_2C_4'.$$

Consequently, the pressure estimate (5.5) is valid where $C_P = 4\,(\frac{C_2\lambda_2C_K}{\lambda_1})^2K_2C_4'$.

• **Saturation estimation.** We multiply the 1st equation of (4.2) by $\beta(S_L^{n+1})$, then we sum over $L\in\mathcal{V}$ and $n\in[\![0,l-1]\!]$. This yields

$$F_1+F_2+F_3+F_4+F_5 = 0, \tag{5.15}$$

where

$$F_1 = \sum_{L\in\mathcal{V}} |\omega_L|\phi_L \sum_{n=0}^{l-1}(S_L^{n+1}-S_L^n)\beta(S_L^{n+1}),$$

$$F_2 = \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \eta_w(;^P)\mathbb{K}_{LM}^\tau[\delta_{LM}^{n+1}\beta(S)]\delta_{LM}^{n+1}P,$$

$$F_3 = \sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} [\eta_wB_o^+](;^T)\mathbb{K}_{LM}^\tau[\delta_{LM}^{n+1}\beta(S)]\delta_{LM}^{n+1}T,$$

$$F_4 = -\sum_{n=0}^{l-1}\delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} [\eta_wB_o^-](;^{-T})\mathbb{K}_{LM}^\tau[\delta_{LM}^{n+1}\beta(S)]\delta_{LM}^{n+1}T,$$

$$F_5 = -\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} [\Lambda_0]_\tau^{n+1} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau [\delta_{LM}^{n+1}\beta(S)](\gamma_{LM}^{n+1}\delta_{LM}^{n+1}S).$$

Let φ be a function such that $\varphi'(S) = \beta(S)$. The accumulation term can be bounded from below with a telescopic series leading to

$$F_1 \geq \sum_{L\in\mathcal{V}} |\omega_L|\phi_L[\varphi(S_L^l) - \varphi(S_L^0)]. \tag{5.16}$$

Moreover, the fact that $\gamma(S) = \beta'(S)$ implies that for $\tau \in \mathcal{T}$ and $L, M \in \mathcal{V}_\tau$:

$$\exists\, S^* \in I_{LM}^{n+1} : \qquad \beta(S_M^{n+1}) - \beta(S_L^{n+1}) = \gamma(S^*)(S_M^{n+1} - S_L^{n+1}). \tag{5.17}$$

Regardless the sign of $\mathbb{K}_{LM}^\tau$, one always has

$$\mathbb{K}_{LM}^\tau \gamma_{LM}^{n+1} \geq \mathbb{K}_{LM}^\tau \gamma(S^*).$$

Thanks to the fact that β is nondecreasing and (5.17), one finds

$$\mathbb{K}_{LM}^\tau \gamma_{LM}^{n+1}[\delta_{LM}^{n+1}\beta(S)]\delta_{LM}^{n+1}S \geq \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2.$$

Because of $\Lambda_0 \geq \Lambda_{0,\min} > 0$, one infers

$$F_4 \geq \Lambda_{0,\min} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2. \tag{5.18}$$

Recall that $0 < \lambda \leq \lambda_2$ and $0 \leq \eta_w \leq 1$. Similarly to the pressure estimation, one computes

$$\begin{aligned} |F_2| &\leq \lambda_2 \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} |\mathbb{K}_{LM}^\tau||\delta_{LM}^{n+1}\beta(S)||\delta_{LM}^{n+1}P| \\ &\leq \frac{\Lambda_{0,\min}}{4} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2 + \frac{(\lambda_2 C_K)^2}{\Lambda_{0,\min}} C_P. \end{aligned} \tag{5.19}$$

In the same fashion for F_3 and F_4, but this time using the fact that $|B_o| \leq C_2$ implies

$$|F_3| + |F_4| \leq \frac{\Lambda_{0,\min}}{4} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2 + 4\,\frac{(\lambda_2 C_K)^2}{\Lambda_{0,\min}} K_2 C_4. \tag{5.20}$$

Now, from (5.15), one has

$$F_1 + F_5 = -F_2 - F_3 - F_4 \leq |F_2| + |F_3| + |F_4|.$$

Therefore, by virtue of inequalities (5.16), (5.18), (5.19) and (5.20), we finally find

$$\sum_{L\in\mathcal{V}} |\omega_L|\phi_L[\varphi(S_L^l) - \varphi(S_L^0)] + \frac{\Lambda_{0,\min}}{2} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2 \leq C_\beta,$$

where $C_\beta = \frac{(\lambda_2 C_K)^2}{\Lambda_{0,\min}}(C_P + 4\,K_2C_4)$. Hence, this establishes the capillary term estimate (5.6). By setting $\varphi(S) = \int_1^S \beta(u)\mathrm{d}u$, one has $0 \leq \varphi \leq \varphi(0) < +\infty$. Thus

$$\varphi(S_L^l) - \varphi(S_L^0) \leq \varphi(0).$$

Additionally, from inequality (5.6), there holds

$$\begin{aligned} \frac{\Lambda_{0,\min}}{2} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau} \mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}\beta(S))^2 &\leq C_\beta + \sum_{L\in\mathcal{V}} |\omega_L|\phi_L[\varphi(S_L^0) - \varphi(S_L^l)] \\ &\leq C_\beta + \varphi(0)|\Omega|, \end{aligned}$$

which yields to the requested estimation. The proof is concluded. ∎

We now state the existence result to the proposed finite volume scheme. The proof follows standard arguments of the literature as done for instance in [21], and is therefore skipped here. It makes use of the monotony criterion characterizing the zeros of vector fields (see [19, p. 529]).

Proposition 5.6. *Under the assumptions on the physical data* (A.1)–(A.11)*, there exists at least one solution* $(S_L^{n+1}, P_L^{n+1}, T_L^{n+1})_{L\in\mathcal{V}, n\in[\![0,l-1]\!]}$ *to the numerical scheme* (4.2).

5.4. Saturation and temperature strong convergence

In this part, we show the strong convergence for some saturation and temperature subsequence. For that purpose, we use Fréchet–Kolmogorov theorem. To apply the theorem, we start by showing some compactness results.

5.4.1. *Space compactness for the finite volume scheme*

We begin by showing some space compactness results for the saturation and temperature finite volume approximation.

Proposition 5.7. *For a primal mesh* $\mathcal{T}$ *of* Ω *and a time discretization* $(t^n)_{n\in[\![0,l]\!]}$ *of* $[0,t_f]$*, let* $(S_L^{n+1}, P_L^{n+1}, T_L^{n+1})_{L\in\mathcal{V}, n\in[\![0,l-1]\!]}$ *be a solution to the associated numerical scheme* (4.2)*. Define* $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h\,\widetilde{S}_{h,\delta t}$ *where* $\widetilde{\phi}_h$ *and* $\widetilde{S}_{h,\delta t}$ *are the porosity and saturation finite volume approximations respectively. For* $y\in\mathbb{R}^d$*, we define*

$$\Omega_y := \{x\in\Omega/[x,x+y]\subset\Omega\}.$$

Under the assumptions (A.1)–(A.11)*, we have*

$$\lim_{|y|,h\to 0^+}\int_{\Omega_y\times(0,t_f)} |\widetilde{u}_{h,\delta t}(x+y,t)-\widetilde{u}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Note that this limit is uniform in δt*. Indeed, it does not depend on the time discretization.*

Proof. Let $y\in\mathbb{R}^d$. One has

$$\begin{aligned}
A^y_{h,\delta t} &:= \int_0^{t_f}\int_{\Omega_y} |\widetilde{u}_{h,\delta t}(x+y,t)-\widetilde{u}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^{t_f}\int_{\Omega_y} |\widetilde{\phi}_h(x+y)\,\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{\phi}_h(x)\,\widetilde{S}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^{t_f}\int_{\Omega_y} |(\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x))\,\widetilde{S}_{h,\delta t}(x+y,t)|\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_0^{t_f}\int_{\Omega_y} |\widetilde{\phi}_h(x)\,(\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{S}_{h,\delta t}(x,t))|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Therefore, noting that $0\le\widetilde{S}_{h,\delta t}\le 1$ and $0\le\widetilde{\phi}_h\le\phi_2$, one gets

$$A^y_{h,\delta t}\le t_f B^y_h + \phi_2 C^y_{h,\delta t}, \tag{5.21}$$

where

$$B^y_h := \int_{\Omega_y} |\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x)|\,\mathrm{d}x;\, C^y_{h,\delta t} := \int_0^{t_f}\int_{\Omega_y} |\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{S}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t.$$

Firstly, we have

$$\begin{aligned}
B^y_h &\le \int_{\Omega_y} |\widetilde{\phi}_h(x+y)-\phi(x+y)|\,\mathrm{d}x + \int_{\Omega_y} |\widetilde{\phi}_h(x)-\phi(x)|\,\mathrm{d}x + \int_{\Omega_y} |\phi(x+y)-\phi(x)|\,\mathrm{d}x\\
&\le 2\int_{\Omega} |\widetilde{\phi}_h(x)-\phi(x)|\,\mathrm{d}x + \int_{\Omega_y} |\phi(x+y)-\phi(x)|\,\mathrm{d}x.
\end{aligned} \tag{5.22}$$

Now, noting that $\phi \in L^\infty(\Omega)$ and that Ω is bounded, one gets $\phi \in L^1(\Omega)$. Thus

$$\lim_{|y|\to 0^+} \int_{\Omega_y} |\phi(x+y) - \phi(x)| \, \mathrm{d}x = 0.$$

Moreover, the fact that $\phi \in L^\infty(\Omega)$ implies that

$$\phi_h \xrightarrow[h\to 0^+]{} \phi \quad \text{a.e. (almost everywhere) on } \Omega,$$

which yields:

$$\lim_{h\to 0^+} \int_{\Omega} |\widetilde{\phi}_h(x) - \phi(x)| \, \mathrm{d}x = 0.$$

Hence, from inequality (5.22), we deduce that

$$\lim_{h\to 0^+} B_h^y = 0. \tag{5.23}$$

Secondly, by virtue of assumption (A.7), one has

$$C_{h,\delta t}^y \le C_\beta \int_0^{t_f} \int_{\Omega_y} |\beta(\widetilde{S}_{h,\delta t}(x+y,t)) - \beta(\widetilde{S}_{h,\delta t}(x,t))|^\theta \, \mathrm{d}x \, \mathrm{d}t.$$

Hölder's inequality implies

$$C_{h,\delta t}^y \le C_\beta (t_f|\Omega|)^{1-\theta} (D_{h,\delta t}^y)^\theta, \tag{5.24}$$

where

$$\begin{aligned} D_{h,\delta t}^y &:= \int_0^{t_f} \int_{\Omega_y} |\beta(\widetilde{S}_{h,\delta t}(x+y,t)) - \beta(\widetilde{S}_{h,\delta t}(x,t))| \mathrm{d}x \, \mathrm{d}t \\ &= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{L\in\mathcal{V}_\tau} \sum_{M\in\mathcal{V}} |\beta(S_M^{n+1}) - \beta(S_L^{n+1})| \mu_{\mathbb{R}^d}(x \in \Omega_y \cap \tau \cap \omega_L / x+y \in \omega_M) \\ &\le \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} |\beta(S_M^{n+1}) - \beta(S_L^{n+1})| \mu_{\mathbb{R}^d}(x \in \Omega_y / \sigma_{LM}^\tau \cap [x, x+y] \neq \emptyset). \end{aligned}$$

Now, note that

$$\mu_{\mathbb{R}^d}(x \in \Omega_y / \sigma_{LM}^\tau \cap [x, x+y] \neq \emptyset) \le C_d^1 |\sigma_{LM}^\tau| |y|,$$

where C_d^1 is a constant depending only on the space dimension d. Hence, one obtains

$$\begin{aligned} D_{h,\delta t}^y &\le C_d^1 |y| \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} |\sigma_{LM}^\tau| |\beta(S_M^{n+1}) - \beta(S_L^{n+1})| \\ &\le C_d^1 |y| \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} |\sigma_{LM}^\tau| h_\tau |\nabla[\beta(S)]_{h,\delta t}^{n+1}{}_{|\tau}|. \end{aligned}$$

Moreover, note that for every $\tau \in \mathcal{T}$ and $\sigma_{LM}^\tau \in \mathcal{E}_\tau$, one has

$$|\sigma_{LM}^\tau| \le C_d^2 (h_\tau)^{d-1} \quad \text{and} \quad |\tau| \ge C_d^3 (\rho_\tau)^d,$$

for some positive constants C_d^2 and C_d^3 depending only on the space dimension d. Therefore

$$|\sigma_{LM}^\tau| h_\tau \le C_d^2 (h_\tau)^d \le C_d^2 \left(\frac{h_\tau}{\rho_\tau}\right)^d (\rho_\tau)^d \le \frac{C_d^2}{C_d^3} \left(\frac{h_\tau}{\rho_\tau}\right)^d |\tau|.$$

And, by virtue of the regularity assumption on the primal mesh (A.9), one gets

$$|\sigma_{LM}^\tau| h_\tau \le \frac{C_d^2}{C_d^3} (C_0)^d |\tau|.$$

Therefore

$$D^y_{h,\delta t} \leq C^4_d |y| \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau| \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} |\nabla[\beta(S)]^{n+1}_{h,\delta t}{}_{|\tau}|$$

$$\leq C^5_d |y| \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau||\nabla[\beta(S)]^{n+1}_{h,\delta t}{}_{|\tau}|.$$

And, in light of Cauchy–Schwarz inequality, one gets

$$D^y_{h,\delta t} \leq C^5_d (t_f|\Omega|)^{\frac{1}{2}} |y| (\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau||\nabla[\beta(S)]^{n+1}_{h,\delta t}{}_{|\tau}|^2)^{\frac{1}{2}}$$

$$\leq C^6_d |y| \|\nabla[\beta(S)]_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d}.$$

And, owing to inequality (5.24), one gets

$$C^y_{h,\delta t} \leq C^7_d |y|^\theta \|\nabla[\beta(S)]_{h,\delta t}\|^\theta_{(L^2(\mathrm{Q}))^d}.$$

Thus, by virtue of the capillary term a priori estimate, there holds

$$C^y_{h,\delta t} \leq C^8_d |y|^\theta. \tag{5.25}$$

Note that $(C^i_d)_{4\leq i\leq 8}$ are some positive constants depending only on the problem data. Finally, it follows from formulas (5.21), (5.23) and (5.25) that

$$\lim_{|y|,h\to 0^+} A^y_{h,\delta t} = 0.$$

This concludes the proof. ■

Proposition 5.8. *For a primal mesh $\mathcal{T}$ of Ω and a time discretization $(t^n)_{n\in[\![0,l]\!]}$ of $[0,t_f]$, let $(S^{n+1}_L, P^{n+1}_L, T^{n+1}_L)_{L\in\mathcal{V},n\in[\![0,l-1]\!]}$ be a solution to the associated numerical scheme (4.2). Define $\widetilde{\psi}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})$ and $\widetilde{u}_{h,\delta t} = \widetilde{\psi}_{h,\delta t}\widetilde{T}_{h,\delta t}$ where $\widetilde{\phi}_h$, $\widetilde{S}_{h,\delta t}$ and $\widetilde{T}_{h,\delta t}$ are the porosity, saturation and temperature finite volume approximations respectively. There holds*

$$\lim_{|y|,h\to 0^+} \int_{\Omega_y\times(0,t_f)} |\widetilde{u}_{h,\delta t}(x+y,t) - \widetilde{u}_{h,\delta t}(x,t)| \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Note that this limit is uniform in δt. Indeed, it does not depend on the time discretization.

Proof. Let $y \in \mathbb{R}^d$. One has

$$A^y_{h,\delta t} := \int_0^{t_f}\int_{\Omega_y} |\widetilde{u}_{h,\delta t}(x+y,t) - \widetilde{u}_{h,\delta t}(x,t)| \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^{t_f}\int_{\Omega_y} |\widetilde{\psi}_{h,\delta t}(x+y,t)\widetilde{T}_{h,\delta t}(x+y,t) - \widetilde{\psi}_{h,\delta t}(x,t)\,\widetilde{T}_{h,\delta t}(x,t)|\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^{t_f}\int_{\Omega_y} |(\widetilde{\psi}_{h,\delta t}(x+y,t) - \widetilde{\psi}_{h,\delta t}(x,t))\widetilde{T}_{h,\delta t}(x+y,t)|\mathrm{d}x\,\mathrm{d}t$$

$$+ \int_0^{t_f}\int_{\Omega_y} |\widetilde{\psi}_{h,\delta t}(x,t)(\widetilde{T}_{h,\delta t}(x+y,t) - \widetilde{T}_{h,\delta t}(x,t))|\mathrm{d}x\,\mathrm{d}t.$$

And, by virtue of the temperature maximum principle, one has $|\widetilde{T}_{h,\delta t}| \leq T_2 := \max\{-T_{\min}, T_{\max}\}$. Moreover $0 \leq \widetilde{\psi}_{h,\delta t} \leq \psi_2$, where $\psi_2 := c_w + c_o + c_s$. Therefore

$$A^y_{h,\delta t} \leq T_2 E^y_{h,\delta t} + \psi_2 F^y_{h,\delta t}, \tag{5.26}$$

where

$$E^y_{h,\delta t} := \int_0^{t_f}\int_{\Omega_y} |\widetilde{\psi}_{h,\delta t}(x+y,t) - \widetilde{\psi}_{h,\delta t}(x,t)|\mathrm{d}x\,\mathrm{d}t; F^y_{h,\delta t} := \int_0^{t_f}\int_{\Omega_y} |\widetilde{T}_{h,\delta t}(x+y,t) - \widetilde{T}_{h,\delta t}(x,t)|\mathrm{d}x\,\mathrm{d}t.$$

Firstly, one writes

$$\begin{aligned}E^y_{h,\delta t} &= \int_0^{t_f}\int_{\Omega_y}\Big|\widetilde{\phi}_h(x+y)[c_w\widetilde{S}_{h,\delta t}(x+y,t)+c_o(1-\widetilde{S}_{h,\delta t}(x+y,t))]+c_s(1-\widetilde{\phi}_h(x+y))\\
&\qquad -\widetilde{\phi}_h(x)[c_w\widetilde{S}_{h,\delta t}(x,t)+c_o(1-\widetilde{S}_{h,\delta t}(x,t))]-c_s(1-\widetilde{\phi}_h(x))\Big|\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^{t_f}\int_{\Omega_y}\Big\{|(c_w-c_o)(\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x))\widetilde{S}_{h,\delta t}(x+y,t)|+|(c_o-c_s)\widetilde{\phi}_h(x)(\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x))|\\
&\qquad +|(c_w-c_o)\widetilde{\phi}_h(x)(\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{S}_{h,\delta t}(x,t))|\Big\}\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

Therefore

$$\begin{aligned}E^y_{h,\delta t} &\le C^1_d\left[\int_{\Omega_y}|\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x)|\,\mathrm{d}x+\int_0^{t_f}\int_{\Omega_y}|\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{S}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t\right]\\
&\le C^1_d(B^y_h+C^y_{h,\delta t}),\end{aligned}\tag{5.27}$$

where

$$B^y_h := \int_{\Omega_y}|\widetilde{\phi}_h(x+y)-\widetilde{\phi}_h(x)|\,\mathrm{d}x\quad C^y_{h,\delta t} := \int_0^{t_f}\int_{\Omega_y}|\widetilde{S}_{h,\delta t}(x+y,t)-\widetilde{S}_{h,\delta t}(x,t)|\,\mathrm{d}x\,\mathrm{d}t,$$

as in the proof of Proposition 5.7. C^1_d is a positive constant depending only on the problem data. Secondly, in the same fashion as in the proof of Proposition 5.7, one has:

$$\begin{aligned}F^y_{h,\delta t} &\le \sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau}|T^{n+1}_M-T^{n+1}_L|\mu_{\mathbb{R}^d}(x\in\Omega_y/\sigma^\tau_{LM}\cap[x,x+y]\ne\emptyset)\\
&\le C^2_d|y|\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}|\tau|\sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau}|\nabla T^{n+1}_{h,\delta t}{}_{|\tau}|\\
&\le C^3_d|y|\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}|\tau||\nabla T^{n+1}_{h,\delta t}{}_{|\tau}|\le C^4_d|y|\|\nabla T_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d}.\end{aligned}$$

And, by virtue of the temperature a priori estimate, there holds:

$$F^y_{h,\delta t}\le C^5_d|y|.\tag{5.28}$$

Note that $(C^i_d)_{2\le i\le 6}$ are some positive constants depending only on the problem data. Finally, it follows from inequalities (5.26), (5.27) and (5.28) that

$$\lim_{|y|,h\to 0^+}A^y_{h,\delta t}=0.$$

This concludes the proof. ∎

5.4.2. *Space compactness for the continuous in time approximations*

For now, let $\widetilde{u}_{h,\delta t}=\widetilde{\phi}_h\,\widetilde{S}_{h,\delta t}$ or $\widetilde{u}_{h,\delta t}=\psi(\widetilde{\phi}_h,\widetilde{S}_{h,\delta t})\,\widetilde{T}_{h,\delta t}$ where $\widetilde{\phi}_h$, $\widetilde{S}_{h,\delta t}$ and $\widetilde{T}_{h,\delta t}$ are the porosity, saturation and temperature finite volume approximations respectively. We define

$$\begin{aligned}\bar{u}_{h,\delta t}(x,t) &:= \sum_{n=0}^{l-1}\frac{(t^{n+1}-t)\widetilde{u}_{h,\delta t}(x,t^n)+(t-t^n)\widetilde{u}_{h,\delta t}(x,t^{n+1})}{\delta t}\mathbb{1}_{[t^n,t^{n+1})}(t)+\widetilde{u}_{h,\delta t}(x,t_f)\mathbb{1}_{[t_f,+\infty)}(t)\\
&:= \sum_{L\in\mathcal{V}}\mathbb{1}_{\omega_L}(x)\sum_{n=0}^{l-1}\frac{(t^{n+1}-t)u^n_L+(t-t^n)u^{n+1}_L}{\delta t}\mathbb{1}_{[t^n,t^{n+1})}(t)+u^l_L\mathbb{1}_{[t_f,+\infty)}(t),\end{aligned}$$

where $u_L^n = \phi_L S_L^n$ if $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h\, \widetilde{S}_{h,\delta t}$ and $u_L^n = \psi(\phi_L, S_L^n) T_L^n$ if $\widetilde{u}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})\widetilde{T}_{h,\delta t}$, for $n \in [\![0, l]\!]$ and $L \in \mathcal{V}$.

Proposition 5.9. *For $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h\, \widetilde{S}_{h,\delta t}$ or $\widetilde{u}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})\, \widetilde{T}_{h,\delta t}$, there holds*

$$\lim_{|y|,h,\delta t \to 0^+} \int_{\Omega_y \times (0,t_f)} |\bar{u}_{h,\delta t}(x+y,t) - \bar{u}_{h,\delta t}(x,t)|\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Proof. One has

$$\begin{aligned} G^y_{h,\delta t} &:= \int_0^{t_f} \int_{\Omega_y} |\bar{u}_{h,\delta t}(x+y,t) - \bar{u}_{h,\delta t}(x,t)|\, \mathrm{d}x\, \mathrm{d}t \\ &:= \sum_{n=0}^{l-1} \int_{t^n}^{t_{n+1}} \int_{\Omega_y} \frac{1}{\delta t} \{(t^{n+1} - t)(\widetilde{u}_h^n(x+y) - \widetilde{u}_h^n(x)) + (t - t^n)(\widetilde{u}_h^{n+1}(x+y) - \widetilde{u}_h^{n+1}(x))\}\, \mathrm{d}x\, \mathrm{d}t \\ &\le \sum_{n=0}^{l-1} \int_{t^n}^{t_{n+1}} \int_{\Omega_y} (|\widetilde{u}_h^n(x+y) - \widetilde{u}_h^n(x)| + |\widetilde{u}_h^{n+1}(x+y) - \widetilde{u}_h^{n+1}(x)|)\, \mathrm{d}x\, \mathrm{d}t \\ &\le 2 \int_0^{t_f} \int_{\Omega_y} |\widetilde{u}_{h,\delta t}(x+y,t) - \widetilde{u}_{h,\delta t}(x,t)|\, \mathrm{d}x\, \mathrm{d}t + \delta t^0 \int_{\Omega_y} |\widetilde{u}_h^0(x+y) - \widetilde{u}_h^0(x)|\, \mathrm{d}x\, \mathrm{d}t \\ &\le 2A^y_{h,\delta t} + B^y_{h,\delta t}, \end{aligned} \tag{5.29}$$

where

$$A_{h,\delta t} = \int_{\Omega_y \times (0,t_f)} |\widetilde{u}_{h,\delta t}(x+y,t) - \widetilde{u}_{h,\delta t}(x,t)|\, \mathrm{d}x\, \mathrm{d}t; B^y_{h,\delta t} = 2\delta t \int_\Omega |\widetilde{u}_h^0(x)|\, \mathrm{d}x.$$

Now, note that $(\widetilde{u}_h^0)_h$ is uniformly bounded. Indeed, for $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h\, \widetilde{S}_{h,\delta t}$, one has: $\widetilde{u}_h^0 = \widetilde{\phi}_h\, \widetilde{S}_h^0$, thus $|\widetilde{u}_h^0| \le \phi_2$. Moreover, for $\widetilde{u}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})\, \widetilde{T}_{h,\delta t}$, one has: $\widetilde{u}_h^0 = \psi(\widetilde{\phi}_h, \widetilde{S}_h^0)\widetilde{T}_h^0$, thus, by virtue of the temperature maximum principle, one gets: $|\widetilde{u}_h^0| \le \psi_2 T_2$, where $\psi_2 = c_w + c_o + c_s$ and $T_2 = \max\{-T_{\min}, T_{\max}\}$. Therefore

$$B^y_{h,\delta t} \le 2\delta t |\Omega| C_d^1 = C_d^2 \delta t,$$

where C_d^1 and C_d^2 are some positive constants depending only on the problem data. Thus, inequality (5.29) becomes

$$G^y_{h,\delta t} \le 2A^y_{h,\delta t} + C_d^2 \delta t.$$

Now, from Propositions 5.7 and 5.8, it follows that

$$\lim_{|y|,h \to 0^+} A^y_{h,\delta t} = 0,$$

uniformly in δt. This yields, as $|y|$, h and δt tend to 0

$$\lim_{|y|,h,\delta t \to 0^+} G^y_{h,\delta t} = 0.$$

This concludes the proof. ■

5.4.3. *Time compactness for the continuous in time approximations*

The proof of this part follows that of Lemma A.1 in [8]. Let $z > 0$. For $t \ge 0$, we define

$$\bar{w}_{h,\delta t}(\,\cdot\,, t) = \bar{u}_{h,\delta t}(\,\cdot\,, t+z) - \bar{u}_{h,\delta t}(\,\cdot\,, t),$$

where $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h \, \widetilde{S}_{h,\delta t}$ or $\widetilde{u}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t}) \, \widetilde{T}_{h,\delta t}$. Note that $\overline{w}_{h,\delta t}(\,\cdot\,,t)$ vanishes for $t \geq t_f$. We introduce the notation:

$$\begin{aligned} J_{h,\delta t}(z) &:= \int_0^{+\infty} \int_\Omega |\overline{u}_{h,\delta t}(x,t+z) - \overline{u}_{h,\delta t}(x,t)| \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^{+\infty} \int_\Omega |\overline{w}_{h,\delta t}(x,t)| \, \mathrm{d}x \, \mathrm{d}t, \end{aligned} \tag{5.30}$$

which is well-defined.

Proposition 5.10. *For $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h \, \widetilde{S}_{h,\delta t}$ or $\widetilde{u}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t}) \, \widetilde{T}_{h,\delta t}$, there holds:*

$$\lim_{z,h,\delta t \to 0^+} J_{h,\delta t}(z) = 0.$$

Note that this limit does not depend on the space or time discretization.

Proposition 5.11. *The sequences of finite volume approximations $(\widetilde{u}_{h,\delta t})_{h,\delta t}$ are relatively compact in $L^1(\mathrm{Q})$.*

Let us end this section by the following remark.

Remark 5.12. For simplified models (advection-diffusion, or for degenerate parabolic problems) in the isothermal case, the convergence of the numerical schemes is proved rigorously either by compactness arguments, or by obtaining a priori error estimates, see for instance [15] and the references therein. In this paper, the convergence makes use of compactness arguments for passing to the limit in nonlinearities. Deriving theoretical error estimates in the nonisothermal case is difficult because the model is too complex. This could be treated in future contributions.

6. Convergence of the numerical scheme

6.1. Strong and weak convergence properties

Lemma 6.1. *Under the assumptions* (A.1)–(A.11)*, let us define $\widetilde{u}_{h,\delta t} = \widetilde{\phi}_h \, \widetilde{S}_{h,\delta t}$ and $u_{h,\delta t} = \widetilde{\phi}_h \, S_{h,\delta t}$. There holds*

$$\lim_{h \to 0} \|u_{h,\delta t} - \widetilde{u}_{h,\delta t}\|_{L^1(\mathrm{Q})} = 0.$$

Proof. It can be checked that

$$A_{h,\delta t} := \|u_{h,\delta t} - \widetilde{u}_{h,\delta t}\|_{L^1(\mathrm{Q})} = \int_\mathrm{Q} |\widetilde{\phi}_h \, S_{h,\delta t} - \widetilde{\phi}_h \, \widetilde{S}_{h,\delta t}| \, \mathrm{d}x \, \mathrm{d}t \leq \phi_2 \int_\mathrm{Q} |S_{h,\delta t} - \widetilde{S}_{h,\delta t}| \, \mathrm{d}x \, \mathrm{d}t.$$

In light of assumption (A.7), one gets

$$A_{h,\delta t} \leq \phi_2 C_\beta \int_\mathrm{Q} |\beta(S_{h,\delta t}) - \beta(\widetilde{S}_{h,\delta t})|^\theta \, \mathrm{d}x \, \mathrm{d}t.$$

Thus, by virtue of Hölder's inequality, one has:

$$A_{h,\delta t} \leq C_d^1 \left(\int_\mathrm{Q} |\beta(S_{h,\delta t}) - \beta(\widetilde{S}_{h,\delta t})| \, \mathrm{d}x \, \mathrm{d}t \right)^\theta \leq C_d^1 (B_{h,\delta t})^\theta, \tag{6.1}$$

where $(C_d^i)_i$ are some positive constants depending only on the problem data, and

$$\begin{aligned}B_{h,\delta t} &:= \|\beta(S_{h,\delta t}) - \beta(\widetilde{S}_{h,\delta t})\|_{L^1(\mathrm{Q})}\\ &= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{L\in\mathcal{V}_\tau} \int_{\omega_L\cap\tau} |\beta(S_{h,\delta t}^{n+1}) - \beta(\widetilde{S}_{h,\delta t}^{n+1})| \,\mathrm{d}x\\ &= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{L\in\mathcal{V}_\tau} \int_{\omega_L\cap\tau} |\beta(S_{h,\delta t}(x,t^{n+1})) - \beta(\widetilde{S}_{h,\delta t}(x_L,t^{n+1}))| \,\mathrm{d}x\\ &\le h\sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\nabla[\beta(S)]_{h,\delta t}^{n+1}{}_{|\tau}| \le h\int_{\mathrm{Q}} |\nabla[\beta(S)]_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By virtue of the Cauchy–Schwarz inequality, one gets:

$$B_{h,\delta t} \le h\,(t_f|\Omega|)^{\frac{1}{2}}\, \|\nabla[\beta(S)]_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d} \le C_d^2\, h. \tag{6.2}$$

Finally, one shows $\lim_{h\to 0} A_{h,\delta t} = 0$. This concludes the proof. ■

Lemma 6.2. *Under the assumptions* (A.1)–(A.11)*, let us define* $\widetilde{v}_{h,\delta t} = \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})\,\widetilde{T}_{h,\delta t}$ *and* $v_{h,\delta t} = \psi(\widetilde{\phi}_h, S_{h,\delta t})\,T_{h,\delta t}$*. There holds*

$$\lim_{h\to 0} \|v_{h,\delta t} - \widetilde{v}_{h,\delta t}\|_{L^1(\mathrm{Q})} = 0.$$

Proof. Let us set

$$C_{h,\delta t} := \|v_{h,\delta t} - \widetilde{v}_{h,\delta t}\|_{L^1(\mathrm{Q})} = \int_{\mathrm{Q}} |\psi(\widetilde{\phi}_h, S_{h,\delta t})\,T_{h,\delta t} - \psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})\,\widetilde{T}_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t.$$

And recalling that $\psi(\phi, S) = [c_w S + c_o(1-S)]\phi + c_s(1-\phi)$, one gets

$$C_{h,\delta t} = \int_{\mathrm{Q}} \Big|[\widetilde{\phi}_h\, c_o + (1-\widetilde{\phi}_h)c_s](T_{h,\delta t} - \widetilde{T}_{h,\delta t}) + \widetilde{\phi}_h(c_w - c_o)(S_{h,\delta t}T_{h,\delta t} - \widetilde{S}_{h,\delta t}\widetilde{T}_{h,\delta t})\Big| \,\mathrm{d}x\,\mathrm{d}t.$$

Using the fact that $0 < \widetilde{\phi}_h < 1$ and the triangle inequality, one writes

$$\begin{aligned}C_{h,\delta t} \le \Bigg[&(c_o + c_s)\int_{\mathrm{Q}} |T_{h,\delta t} - \widetilde{T}_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t + (c_w + c_o)\int_{\mathrm{Q}} \widetilde{\phi}_h |S_{h,\delta t} - \widetilde{S}_{h,\delta t}||T_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t\\ &+ (c_w + c_o)\int_{\mathrm{Q}} \widetilde{\phi}_h\, \widetilde{S}_{h,\delta t}|T_{h,\delta t} - \widetilde{T}_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t\Bigg].\end{aligned}$$

Therefore, owing to the saturation and temperature maximum principles, one obtains

$$C_{h,\delta t} \le C_d^4(A_{h,\delta t} + D_{h,\delta t}), \tag{6.3}$$

where

$$A_{h,\delta t} = \int_{\mathrm{Q}} \widetilde{\phi}_h |S_{h,\delta t} - \widetilde{S}_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t = \|u_{h,\delta t} - \widetilde{u}_{h,\delta t}\|_{L^1(\mathrm{Q})},$$

and

$$D_{h,\delta t} = \int_{\mathrm{Q}} |T_{h,\delta t} - \widetilde{T}_{h,\delta t}| \,\mathrm{d}x\,\mathrm{d}t.$$

Now, from Lemma 6.1, there holds

$$\lim_{h\to 0} A_{h,\delta t} = 0. \tag{6.4}$$

As a result

$$D_{h,\delta t}=\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{L\in\mathcal{V}_\tau}\int_{\omega_L\cap\tau}|T_{h,\delta t}(x,t^{n+1})-\widetilde{T}_{h,\delta t}(x_L,t^{n+1})|\,\mathrm{d}x$$
$$\leq h\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}|\tau|\,|\nabla T^{n+1}_{h,\delta t}{}_{|\tau}|\leq h\int_{\mathrm{Q}}|\nabla T_{h,\delta t}|\,\mathrm{d}x\,\mathrm{d}t.$$

By virtue of Cauchy–Schwarz inequality, one infers

$$D_{h,\delta t}\leq h\,(t_f|\Omega|)^{\frac{1}{2}}\,\|\nabla T_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d}\leq C^5_d\,h.$$

Thus

$$\lim_{h\to 0}A_{h,\delta t}=0. \tag{6.5}$$

Finally, from (6.3), (6.4) and (6.5), we deduce that

$$\lim_{h\to 0}C_{h,\delta t}=0.$$

This concludes the proof. ■

Proposition 6.3. *Under the assumptions* (A.1)–(A.11)*, there exists a subsequence of* $(S_{h,\delta t},P_{h,\delta t},T_{h,\delta t},\widetilde{S}_{h,\delta t},\widetilde{P}_{h,\delta t},\widetilde{T}_{h,\delta t})$ *such that the following convergence properties hold*

$$\widetilde{u}_{h,\delta t},u_{h,\delta t}\longrightarrow u \text{ strongly in } L^r(\mathrm{Q}),\ \text{for } r\geq 1 \text{ and a.e. on } \mathrm{Q}. \tag{6.6}$$
$$\widetilde{v}_{h,\delta t},v_{h,\delta t}\longrightarrow v \text{ strongly in } L^r(\mathrm{Q}),\ \text{for } r\geq 1 \text{ and a.e. on } \mathrm{Q}. \tag{6.7}$$
$$\widetilde{S}_{h,\delta t},S_{h,\delta t}\longrightarrow S \text{ a.e. on } \mathrm{Q}, \tag{6.8}$$
$$\widetilde{T}_{h,\delta t},T_{h,\delta t}\longrightarrow T \text{ a.e. on } \mathrm{Q}, \tag{6.9}$$
$$P_{h,\delta t}\rightharpoonup P \text{weakly in } L^2(\mathrm{Q}), \tag{6.10}$$
$$\nabla P_{h,\delta t}\rightharpoonup \nabla P \text{ weakly in } (L^2(\mathrm{Q}))^d, \tag{6.11}$$
$$\nabla[\beta(S)]_{h,\delta t}\rightharpoonup \nabla\beta(S) \text{ weakly in } (L^2(\mathrm{Q}))^d, \tag{6.12}$$
$$\nabla T_{h,\delta t}\rightharpoonup \nabla T \text{ weakly in } (L^2(\mathrm{Q}))^d. \tag{6.13}$$

Furthermore, $T,\beta(S),P\in L^2(0,t_f;H^1_{\Gamma_D}(\Omega))$ *satisfy the following:*

$$0\leq S\leq 1 \text{a.e. on } \mathrm{Q}, \tag{6.14}$$
$$T_{\min}\leq T\leq T_{\max} \text{a.e. on } \mathrm{Q}, \tag{6.15}$$
$$u=\phi\,S \text{a.e. on } \mathrm{Q}, \tag{6.16}$$
$$v=\psi(\phi,S)\,T \text{a.e. on } \mathrm{Q}. \tag{6.17}$$

Proof. By virtue of Proposition 5.11, $(\widetilde{u}_{h,\delta t})_{h,\delta t}$ and $(\widetilde{v}_{h,\delta t})_{h,\delta t}$ are relatively compact in $L^1(\mathrm{Q})$. Thus, the following strong convergence properties hold for some subsequence of $(\widetilde{S}_{h,\delta t},\widetilde{P}_{h,\delta t},\widetilde{T}_{h,\delta t})$

$$\widetilde{u}_{h,\delta t}\longrightarrow u \text{ and } \widetilde{v}_{h,\delta t}\longrightarrow v \text{ strongly in } L^1(\mathrm{Q}) \text{ and a.e. on } \mathrm{Q},$$

Moreover, from Lemmas 6.1 and 6.2, one gets

$$u_{h,\delta t}\longrightarrow u \text{ and } v_{h,\delta t}\longrightarrow v \text{ strongly in } L^1(\mathrm{Q}) \text{ and a.e. on } \mathrm{Q},$$

Furthermore, the fact that $(u_{h,\delta t})_{h,\delta t}$ and $(v_{h,\delta t})_{h,\delta t}$ are bounded yields

$$\widetilde{u}_{h,\delta t},u_{h,\delta t}\longrightarrow u \text{ and } \widetilde{v}_{h,\delta t},v_{h,\delta t}\longrightarrow v \text{ strongly in } L^r(\mathrm{Q}),\ \text{for } r\geq 1 \text{ and a.e. on } \mathrm{Q}.$$

Now, noting that $\widetilde{\phi}_h \to \phi$ a.e. on Ω (because $\phi \in L^\infty(\Omega)$), $0 < \phi_1 \le \widetilde{\phi}_h$ and $0 < \phi_1 \le \phi$ a.e. on Ω, one has

$$\widetilde{S}_{h,\delta t} = \frac{\widetilde{u}_{h,\delta t}}{\widetilde{\phi}_h} \longrightarrow \frac{u}{\phi} \quad \text{a.e. on Q,} \qquad \text{and} \qquad S_{h,\delta t} = \frac{u_{h,\delta t}}{\widetilde{\phi}_h} \longrightarrow \frac{u}{\phi} \quad \text{a.e. on Q.}$$

Therefore, by defining $S = u/\phi$ a.e. on Q, one obtains

$$\widetilde{S}_{h,\delta t}, S_{h,\delta t} \quad \longrightarrow \quad S \qquad \text{a.e. on Q.} \tag{6.18}$$

The saturation maximum principle implies $0 \le S \le 1$ a.e. on Q. Moreover, $u = \phi\, S$ a.e. on Q.
For now, recall that $\psi(\phi, S) = [c_w S + c_o(1-S)]\phi + c_s(1-\phi)$. Noting that $\widetilde{\phi}_h \to \phi$ a.e. on Ω and $\widetilde{S}_{h,\delta t} \to S$ a.e. on Q, one infers $\psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t}) \to \psi(\phi, S)$ a.e. on Q. In addition, there exists a constant $\psi_1 > 0$ such that $\psi_1 \le \psi$, thus

$$\widetilde{T}_{h,\delta t} = \frac{\widetilde{v}_{h,\delta t}}{\psi(\widetilde{\phi}_h, \widetilde{S}_{h,\delta t})} \quad \longrightarrow \quad \frac{v}{\psi(\phi, S)} \qquad \text{a.e. on Q.}$$

In the same fashion, one shows that

$$T_{h,\delta t} \quad \longrightarrow \quad \frac{v}{\psi(\phi, S)} \qquad \text{a.e. on Q.}$$

Therefore, by defining $T = \frac{v}{\psi(\phi,S)}$ a.e. on Q, one has:

$$\widetilde{T}_{h,\delta t}, T_{h,\delta t} \quad \longrightarrow \quad T \qquad \text{a.e. on Q.}$$

Owing to the temperature maximum principle, one deduces $T_{\min} \le T \le T_{\max}$ a.e. on Q. Furthermore, $v = \psi(\phi, S)\, T$ a.e. on Q.
In light of Proposition 5.5, the sequence $(\nabla P_{h,\delta t})$ is bounded in $(L^2(\mathrm{Q}))^d$. Furthermore, by virtue of the Poincaré inequality, we deduce that the sequence $(P_{h,\delta t})$ is bounded in $L^2(\mathrm{Q})$. Therefore, owing to the fact that $(P_{h,\delta t}) \subset L^2(0, t_f; H^1_{\Gamma_D}(\Omega))$, there exists $P \in L^2(0, t_f; H^1_{\Gamma_D}(\Omega))$ such that the following convergence properties hold for some subsequence of $(P_{h,\delta t})$:

$$P_{h,\delta t} \rightharpoonup P \text{weakly in } L^2(\mathrm{Q}), \quad \text{and} \quad \nabla P_{h,\delta t} \rightharpoonup \nabla P \text{weakly in } (L^2(\mathrm{Q}))^d.$$

Analogously, one proves that there exists $T^* \in L^2(0, t_f; H^1_{\Gamma_D}(\Omega))$ such that the following convergence properties hold for a subsequence of $(T_{h,\delta t})$

$$T_{h,\delta t} \rightharpoonup T^* \text{weakly in } L^2(\mathrm{Q}), \quad \text{and} \quad \nabla T_{h,\delta t} \rightharpoonup \nabla T^* \text{weakly in } (L^2(\mathrm{Q}))^d.$$

Now, note that $T_{h,\delta t} \to T$ a.e. on Q. Thus, by virtue of the dominated convergence theorem together with the temperature maximum principle, we deduce that $T_{h,\delta t} \to T$ strongly in $L^2(\mathrm{Q})$. Therefore, by identifying the limits, one gets $T^* = T$ a.e. on Q.
Similarly, from Proposition 5.5, the sequence $(\nabla[\beta(S)]_{h,\delta t})$ is bounded in $(L^2(\mathrm{Q}))^d$ and the sequence $([\beta(S)]_{h,\delta t})$ is bounded in $L^2(\mathrm{Q})$. Therefore, owing to the fact that $([\beta(S)]_{h,\delta t}) \subset L^2(0, t_f; H^1_{\Gamma_D}(\Omega))$ (because $S_{h,\delta t} = P_c^{-1}(P_o - P_w) = P_c^{-1}(0) = 1$ over $\Gamma_D \times (0, t_f)$ and $\beta(1) = 0$ by the definition of β), there exists $\beta^* \in L^2(0, t_f; H^1_{\Gamma_D}(\Omega))$ and $\zeta \in (L^2(\mathrm{Q}))^d$ such that, up to a subsequence, one infers

$$[\beta(S)]_{h,\delta t} \rightharpoonup \beta^* \text{weakly in } L^2(\mathrm{Q}), \quad \text{and} \quad \nabla[\beta(S)]_{h,\delta t} \rightharpoonup \zeta \text{weakly in } (L^2(\mathrm{Q}))^d.$$

Moreover, combining (6.18) together with the continuity of β imply

$$\beta(\widetilde{S}_{h,\delta t}) \quad \longrightarrow \quad \beta(S) \qquad \text{a.e. on Q.}$$

And, in view of the saturation maximum principle, $(\beta(\widetilde{S}_{h,\delta t}))$ is bounded. Therefore, by virtue of the dominated convergence theorem, one gets

$$\beta(\widetilde{S}_{h,\delta t}) \quad \longrightarrow \quad \beta(S) \qquad \text{strongly in } L^2(\mathrm{Q}). \tag{6.19}$$

Mimicking the proof of Lemma 6.2, it can be established that

$$\begin{aligned} \|[\beta(S)]_{h,\delta t} - \beta(\widetilde{S}_{h,\delta t})\|_{L^2(\mathrm{Q})} &\le h\, C_d^6\, \|\nabla[\beta(S)]_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d} \\ &\le C_d^7\, h, \end{aligned}$$

where $(C_d^i)_i$ are some positive constants depending only on the problem data. Thus

$$\lim_{h\to 0} \|[\beta(S)]_{h,\delta t} - \beta(\widetilde{S}_{h,\delta t})\|_{L^2(\mathrm{Q})} = 0. \tag{6.20}$$

Therefore, from (6.19) and (6.20), one gets:

$$[\beta(S)]_{h,\delta t} \longrightarrow \beta(S) \qquad \text{strongly in } L^2(\mathrm{Q}).$$

The limit uniqueness forces $\beta^* = \beta(S)$ a.e. on Q. Hence, by identifying the limits, we deduce that $\zeta = \nabla\beta^* = \nabla\beta(S)$ a.e. on Q. As a result $\beta(S) \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega))$. The proof is concluded. ■

6.2. Weak solution to the continuous problem

Definition 6.4 (Weak solution)**.** Let S^0 and T^0 be two functions in $L^\infty(\Omega)$ such that $0 \le S^0(x) \le 1$ and $T_{\min} \le T^0(x) \le T_{\max}$ a.e. for $x \in \Omega$. We say that (S,P,T) is a weak solution to the problem (2.1), if it satisfies: $0 \le S \le 1, T_{\min} \le T \le T_{\max}$ and $\beta(S), P, T \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega))$. Moreover, for every $\xi \in C_c^\infty(\overline{\Omega}\times[0,t_f))$ such that $\xi(x,t) = 0$ for $(x,t) \in \Gamma_D \times [0,t_f)$, there holds:

$$\begin{aligned}
-\int_{\mathrm{Q}} \phi\, S\, \partial_t\xi \,\mathrm{d}x\,\mathrm{d} - \int_\Omega \phi\, S^0\, \xi(x,0)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathrm{Q}} \lambda(S;T)\,\eta_w(S;T)\,(\mathbb{K}\nabla P)\cdot\nabla\xi \,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\mathrm{Q}} \lambda(S;T)\,\eta_w(S;T)\,B_o(S;T)\,(\mathbb{K}\nabla T)\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\mathrm{Q}} \Lambda_0(S;T)\,(\mathbb{K}\nabla\beta(S))\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned} \tag{6.21}$$

$$\begin{aligned}
\int_{\mathrm{Q}} \phi\, S\, \partial_t\xi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \phi\, S^0\, \xi(x,0)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathrm{Q}} \lambda(S;T)\,\eta_o(S;T)\,(\mathbb{K}\nabla P)\cdot\nabla\xi \,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\mathrm{Q}} \lambda(S;T)\,\eta_o(S;T)\,B_o(S;T)\,(\mathbb{K}\nabla T)\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\
- \int_{\mathrm{Q}} \Lambda_0(S;T)\,(\mathbb{K}\nabla\beta(S))\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
-\int_{\mathrm{Q}} \psi(\phi;S)\,T\,\partial_t\xi\,\mathrm{d}x\mathrm{d}t - \int_\Omega \psi(\phi;S^0)\,T^0\,\xi(x,0)\,\mathrm{d}x\mathrm{d}t \\
+ \int_{\mathrm{Q}} [\lambda(c_w\eta_w + c_o\eta_o)](S;T)T(\mathbb{K}\nabla P)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t - \int_{\mathrm{Q}} \Lambda_0(S;T)\,(c_o - c_w)\,T\,(\mathbb{K}\nabla\beta(S))\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t \\
+ \int_{\mathrm{Q}} [\lambda(c_w\eta_w + c_o\eta_o)B_o](S;T)T(\mathbb{K}\nabla T)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t + \int_{\mathrm{Q}} (\kappa_T\nabla T)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t = 0.
\end{aligned} \tag{6.23}$$

6.3. Theorem of convergence towards a weak solution

Let us introduce the functions $\underline{w}_{h,\delta t}$, $\overline{w}_{h,\delta t}$ defined a.e. on Q for all $\tau \in \mathcal{T}$ and $n \in [\![0,l-1]\!]$ by

$$\underline{w}_{h,\delta t}|_{\tau\times(t^n,t^{n+1}]} = \underline{w}_\tau^{n+1} := \inf_{x\in\tau} w_{h,\delta t}(x,t^{n+1}) = \min_{M\in\mathcal{V}_\tau} w_M^{n+1},$$

$$\overline{w}_{h,\delta t}|_{\tau\times(t^n,t^{n+1}]} = \overline{w}_\tau^{n+1} := \sup_{x\in\tau} w_{h,\delta t}(x,t^{n+1}) = \max_{M\in\mathcal{V}_\tau} w_M^{n+1},$$

Typically, the function w is either the saturation or the temperature. These upper and lower sequences converge to the same limit. This is the object of the following result.

Lemma 6.5. *There holds*

$$\lim_{h\to 0} \|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|_{L^2(\mathrm{Q})} = 0 \quad \textit{and} \quad \lim_{h\to 0} \|\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}\|_{L^2(\mathrm{Q})} = 0.$$

Proof. Firstly, one has

$$\begin{aligned}\|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|^2_{L^2(\mathrm{Q})} &= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau|^2 \\ &\leq \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\beta(\overline{S}^{n+1}_\tau) - \beta(\underline{S}^{n+1}_\tau)|^{2\theta}.\end{aligned}$$

Now, by virtue of the mean value theorem, for every $\tau \in \mathcal{T}$ and $n \in [\![0, l-1]\!]$, there holds:

$$|\beta(\overline{S}^{n+1}_\tau) - \beta(\underline{S}^{n+1}_\tau)| \leq h |\nabla[\beta(S)]^{n+1}_{h,\delta t}{}_{|_\tau}|.$$

Thus

$$\begin{aligned}\|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|^2_{L^2(\mathrm{Q})} &\leq h^{2\theta} \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\nabla[\beta(S)]^{n+1}_{h,\delta t}{}_{|_\tau}|^{2\theta} \\ &\leq h^{2\theta} \int_{\mathrm{Q}} |\nabla[\beta(S)]_{h,\delta t}|^{2\theta}\, \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Owing to Hölder's inequality, one gets

$$\|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|^2_{L^2(\mathrm{Q})} \leq h^{2\theta} (t_f|\Omega|)^{1-\theta} \|\nabla[\beta(S)]_{h,\delta t}\|^{2\theta}_{(L^2(\mathrm{Q}))^d}.$$

Moreover, by virtue of the capillary term a priori estimate, one gets

$$\|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|_{L^2(\mathrm{Q})} \leq C^1_d h^\theta,$$

where $(C^i_d)_i$ are some positive constants depending only on the problem data. Hence

$$\lim_{h\to 0} \|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|_{L^2(\mathrm{Q})} = 0.$$

Secondly, one writes

$$\begin{aligned}\|\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}\|^2_{L^2(\mathrm{Q})} &= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|^2 \\ &\leq h^2 \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} |\tau|\, |\nabla T^{n+1}_{h,\delta t}{}_{|_\tau}|^2 \leq h^2 \|\nabla T_{h,\delta t}\|^2_{(L^2(\mathrm{Q}))^d}.\end{aligned}$$

By virtue of the temperature a priori estimate, one has

$$\|\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}\|^2_{L^2(\mathrm{Q})} \leq C^2_d h.$$

Thus

$$\lim_{h\to 0} \|\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}\|_{L^2(\mathrm{Q})} = 0.$$

The proof is complete. ■

In the sequel, we show the main result of this article, which states that any limit of the discrete solutions is a weak solution to the continuous problem.

Theorem 6.6 (Passage to the limit)**.** *Under the assumptions of Proposition 6.3, the limit function* (S, P, T) *given in* (6.8)–(6.9)*, is a weak solution to the problem* (2.1) *in the sense of Definition 6.4.*

Proof. We divide the proof in three parts.

(a) Mass conservation equations. We will detail the proof in the case of the mass conservation equation for the wetting phase. The mass conservation equation of the nonwetting phase is carried out similarly. For that purpose, let $\xi \in C^\infty_c(\overline{\Omega} \times [0, t_f))$ be such that $\xi(x, t) = 0$ for $(x, t) \in \Gamma_D \times [0, t_f)$.

We start by multiplying the 1st equation of the numerical scheme (4.2) by $\delta t\, \xi_L^n$ where $\xi_L^n := \xi(x_L, t^n)$ for $L \in \mathcal{V} \setminus \mathcal{V}_D$ (where $\mathcal{V}_D$ is the set of the Dirichlet boundary vertices) and $n \in [\![0, l-1]\!]$, then we sum over L and n. This yields

$$W_1^{h,\delta t} + W_2^{h,\delta t} + W_3^{h,\delta t} + W_4^{h,\delta t} + W_5^{h,\delta t} + W_6^{h,\delta t} = 0, \tag{6.24}$$

where

$$W_1^{h,\delta t} = \sum_{n=0}^{l-1} \sum_{L\in\mathcal{V}} |\omega_L| \phi_L (S_L^{n+1} - S_L^n) \xi_L^n,$$

$$W_2^{h,\delta t} = -\sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1} \mathbb{K}_{LM}^\tau [\beta(S_M^{n+1}) - \beta(S_L^{n+1})] \cdot \xi_L^n,$$

$$W_3^{h,\delta t} = -\sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1} \mathbb{K}_{LM}^\tau \{\gamma_{LM}^{n+1}(S_M^{n+1} - S_L^{n+1}) - [\beta(S_M^{n+1}) - \beta(S_L^{n+1})]\} \cdot \xi_L^n,$$

$$W_4^{h,\delta t} = -\sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} \eta_w(;^P) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} P \cdot \xi_L^n,$$

$$W_5^{h,\delta t} = -\sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} [\eta_w B_o^+](;^T) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T \cdot \xi_L^n,$$

$$W_6^{h,\delta t} = \sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} \lambda_\tau^{n+1} [\eta_w B_o^-](;^{-T}) \mathbb{K}_{LM}^\tau \delta_{LM}^{n+1} T \cdot \xi_L^n.$$

First, rearranging the accumulation term $W_1^{h,\delta t}$ and taking into account that $\xi_L^l = \xi(x_L, t_f) = 0$, one gets

$$\begin{aligned} W_1^{h,\delta t} &= -\sum_{n=0}^{l-1} \sum_{L\in\mathcal{V}} \int_{t^n}^{t_{n+1}} \int_{\omega_L} \phi_L\, S_L^{n+1} \partial_t \xi(x_L, t)\, \mathrm{d}x\mathrm{d}t - \sum_{L\in\mathcal{V}} \int_{\omega_L} \phi_L\, S_L^0\, \xi(x_L, 0)\, \mathrm{d}x \\ &= -\int_{\mathrm{Q}} \widetilde{\phi}_h \widetilde{S}_{h,\delta t} \widetilde{\zeta}_h^1(x,t)\, \mathrm{d}x\mathrm{d}t - \int_\Omega \widetilde{\phi}_h \widetilde{S}_h^0 \widetilde{\xi}_h^0\, \mathrm{d}x, \end{aligned}$$

where

$$\widetilde{\zeta}_h^1(x,t) := \partial_t \xi(x_L, t) \qquad \text{for } x \in \omega_L \text{ and } t \in [0, t_f),$$

$$\widetilde{S}_h^0(x) := S_L^0, \quad \widetilde{\xi}_h^0(x) := \xi(x_L, 0) \qquad \text{for } x \in \omega_L.$$

And, noting that $\xi \in C_c^\infty(\Omega \times [0, t_f))$ and $S^0 \in L^\infty(\Omega)$, one infers

$$\widetilde{\zeta}_h^1 \longrightarrow \partial_t \xi \quad \text{a.e. on Q}, \qquad \widetilde{S}_h^0 \longrightarrow S^0, \quad \widetilde{\xi}_h^0 \longrightarrow \xi(\,\cdot\,, 0) \quad \text{a.e. on } \Omega.$$

From Proposition 6.3, it follows that

$$\widetilde{S}_{h,\delta t} \quad \longrightarrow \quad S \qquad \text{a.e. on Q}.$$

Therefore, by virtue of the dominated convergence theorem together with the fact that all involved sequences of functions are bounded, one has

$$\lim_{h,\delta t\to 0} W_1^{h,\delta t} = -\int_{\mathrm{Q}} \phi S \partial_t \xi\, \mathrm{d}x\mathrm{d}t - \int_\Omega \phi S^0 \xi(x, 0)\, \mathrm{d}x. \tag{6.25}$$

Next, let us show

$$\lim_{h,\delta t\to 0} W_2^{h,\delta t} = \int_{\mathrm{Q}} \Lambda_0(S;T)\, (\mathbb{K}\nabla\beta(S)) \cdot \nabla\xi\, \mathrm{d}x\mathrm{d}t.$$

The discrete integration by part leads to

$$W_2^{h,\delta t} = \sum_{n=0}^{l-1} \delta t \sum_{\tau \in \mathcal{T}} \sum_{\sigma_{LM}^\tau \in \mathcal{E}_\tau} [\Lambda_0]_\tau^{n+1} \mathbb{K}_{LM}^\tau [\beta(S_M^{n+1}) - \beta(S_L^{n+1})] \cdot (\xi_M^n - \xi_L^n).$$

Introducing the following finite element reconstruction of the function ξ

$$\xi_{h,\delta t}^b(x,t) := \sum_{n=0}^{l-1} \sum_{L \in \mathcal{V}} \xi(x_L, \mathbf{t^n}) \varphi_L(x) \mathbb{1}_{(t^n,t^{n+1}]}(t), \qquad \forall\, (x,t) \in \Omega \times (0,t_f),$$

implies

$$W_2^{h,\delta t} = \sum_{n=0}^{l-1} \delta t \sum_{\tau \in \mathcal{T}} [\Lambda_0]_\tau^{n+1} \int_\tau (\mathbb{K}\nabla[\beta(S)]_{h,\delta t}^{n+1}) \cdot \nabla \xi_{h,\delta t}^b(x,t^{n+1})\, \mathrm{d}x.$$

In the sequel, we denote by

$$\begin{aligned} V_2^{h,\delta t} &:= \int_{\mathrm{Q}} \Lambda_0(\underline{S}_{h,\delta t}; \underline{T}_{h,\delta t})\, (\mathbb{K}\nabla[\beta(S)]_{h,\delta t}) \cdot \nabla \xi_{h,\delta t}^b \,\mathrm{d}x\mathrm{d}t \\ &= \sum_{n=0}^{l-1} \delta t \sum_{\tau \in \mathcal{T}} \Lambda_0(\underline{S}_\tau^{n+1}; \underline{T}_\tau^{n+1}) \int_\tau (\mathbb{K}\nabla[\beta(S)]_{h,\delta t}^{n+1}) \cdot \nabla \xi_{h,\delta t}^b(x,t^{n+1})\,\mathrm{d}x. \end{aligned}$$

Now, the fact that Λ_0 is continuous in S and T yields its uniform continuity on the compact set $[0,1] \times [T_{\min}, T_{\max}]$. Hence, there exists a modulus of continuity $\epsilon_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $S_1, S_2 \in [0,1]$ and every $T_1, T_2 \in [T_{\min}, T_{\max}]$, there holds

$$|\Lambda_0(S_1,T_1) - \Lambda_0(S_2,T_2)| \le \epsilon_1(|S_1 - S_2| + |T_1 - T_2|). \tag{6.26}$$

Moreover, ϵ_1 is bounded and

$$\lim_{y \to 0} \epsilon_1(y) = 0.$$

Recall that for every $\tau \in \mathcal{T}$ and $n \in [\![0, l-1]\!]$, one has

$$[\Lambda_0]_\tau^{n+1} = \frac{1}{\#\mathcal{V}_\tau} \sum_{M \in \mathcal{V}_\tau} \Lambda_0(S_M^{n+1}; T_M^{n+1}).$$

Thus

$$[\Lambda_0]_\tau^{n+1} - \Lambda_0(\underline{S}_\tau^{n+1}; \underline{T}_\tau^{n+1}) = \frac{1}{\#\mathcal{V}_\tau} \sum_{M \in \mathcal{V}_\tau} [\Lambda_0(S_M^{n+1}; T_M^{n+1}) - \Lambda_0(\underline{S}_\tau^{n+1}; \underline{T}_\tau^{n+1})].$$

Therefore, by virtue of inequality (6.26) together with the triangle inequality, one finds

$$|[\Lambda_0]_\tau^{n+1} - \Lambda_0(\underline{S}_\tau^{n+1}; \underline{T}_\tau^{n+1})| \le \epsilon_1(|\overline{S}_\tau^{n+1} - \underline{S}_\tau^{n+1}| + |\overline{T}_\tau^{n+1} - \underline{T}_\tau^{n+1}|).$$

Thus

$$\begin{aligned} A_2^{h,\delta t} &:= |W_2^{h,\delta t} - V_2^{h,\delta t}| \\ &\le \sum_{n=0}^{l-1} \delta t \sum_{\tau \in \mathcal{T}} \int_\tau |[\Lambda_0]_\tau^{n+1} - \Lambda_0(\underline{S}_\tau^{n+1}; \underline{T}_\tau^{n+1})|\, |(\mathbb{K}\nabla[\beta(S)]_{h,\delta t}^{n+1}) \cdot \nabla \xi_{h,\delta t}^b(x,t^{n+1})|\,\mathrm{d}x \\ &\le \int_{\mathrm{Q}} \epsilon_1^{h,\delta t} |(\mathbb{K}\nabla[\beta(S)]_{h,\delta t}).\nabla \xi_{h,\delta t}^b|\,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where

$$\epsilon_1^{h,\delta t} := \epsilon_1(|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}| + |\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}|).$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned} A_2^{h,\delta t} &\le \int_{\mathrm{Q}} \epsilon_1^{h,\delta t} |(\mathbb{K}\nabla[\beta(S)]_{h,\delta t}).\nabla[\beta(S)]_{h,\delta t}|^{\frac{1}{2}} |(\mathbb{K}\nabla\xi^b_{h,\delta t})\cdot\nabla\xi^b_{h,\delta t}|^{\frac{1}{2}}\, \mathrm{d}x\mathrm{d}t \\ &\le K_2 \int_{\mathrm{Q}} \epsilon_1^{h,\delta t} |\nabla[\beta(S)]_{h,\delta t}||\nabla\xi^b_{h,\delta t}|\, \mathrm{d}x\mathrm{d}t \\ &\le K_2 \|\nabla\xi^b_{h,\delta t}\|_\infty \|\nabla[\beta(S)]_{h,\delta t}\|_{(L^2(\mathrm{Q}))^d} \int_{\mathrm{Q}} (\epsilon_1^{h,\delta t})^2\, \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Therefore

$$A_2^{h,\delta t} \le C_d^1 \|\nabla\xi^b_{h,\delta t}\|_\infty \int_{\mathrm{Q}} (\epsilon_1^{h,\delta t})^2\, \mathrm{d}x\mathrm{d}t, \tag{6.27}$$

where $(C_d^i)_i$ are some positive constants depending only on the problem data. The regularity of the test function $\xi^b_{h,\delta t}$ entails

$$\lim_{h,\delta t\to 0} \|\nabla\xi^b_{h,\delta t} - \nabla\xi\|_\infty = 0.$$

Thus

$$\lim_{h,\delta t\to 0} \|\nabla\xi^b_{h,\delta t}\|_\infty = \|\nabla\xi\|_\infty < +\infty. \tag{6.28}$$

Now, Lemma 6.5 ensures that

$$\epsilon_1^{h,\delta t} = \epsilon_1(|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}| + |\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}|) \longrightarrow 0 \qquad \text{a.e. on Q.}$$

As a consequence

$$\lim_{h,\delta t\to 0} \int_{\mathrm{Q}} (\epsilon_1^{h,\delta t})^2\, \mathrm{d}x\mathrm{d}t = 0. \tag{6.29}$$

From (6.27), (6.28) and (6.29), we deduce that

$$\lim_{h,\delta t\to 0} A_2^{h,\delta t} = 0.$$

Hence

$$\lim_{h,\delta t\to 0} |W_2^{h,\delta t} - V_2^{h,\delta t}| = 0. \tag{6.30}$$

It remains to show that

$$\lim_{h,\delta t\to 0} V_2^{h,\delta t} = \int_{\mathrm{Q}} \Lambda_0(S;T)\,(\mathbb{K}\nabla\beta(S))\cdot\nabla\xi\, \mathrm{d}x\mathrm{d}t.$$

Notice that

$$\|S_{h,\delta t} - \underline{S}_{h,\delta t}\|_{L^2(\mathrm{Q})} \le \|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}\|_{L^2(\mathrm{Q})} \;\longrightarrow\; 0, \text{ as } h \longrightarrow 0.$$

The fact that $S_{h,\delta t} \to S$ a.e. on Q yields $\underline{S}_{h,\delta t} \to S$ a.e. on Q. Similarly, one shows that $\overline{S}_{h,\delta t} \to S$ a.e. on Q and $\underline{T}_{h,\delta t}, \overline{T}_{h,\delta t} \to T$ a.e. on Q for some subsequences. Therefore, the continuity of Λ_0 yields

$$\Lambda_0(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t}) \;\longrightarrow\; \Lambda_0(S,T) \qquad \text{a.e. on Q.}$$

Thus, owing to the fact that Λ_0 is bounded together with the dominated convergence theorem,, one gets

$$\Lambda_0(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t}) \;\longrightarrow\; \Lambda_0(S,T) \qquad \text{strongly in } L^2(\mathrm{Q}).$$

Moreover, one has

$$\nabla\xi^b_{h,\delta t} \;\longrightarrow\; \nabla\xi \qquad \text{strongly in } (L^2(\mathrm{Q}))^d.$$

Therefore, the fact that $\mathbb{K} \in (L^\infty(\Omega))^{d\times d}$ yields

$$\Lambda_0(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t})\mathbb{K}\nabla\xi^b_{h,\delta t} \;\longrightarrow\; \Lambda_0(S,T)\mathbb{K}\nabla\xi \qquad \text{strongly in } (L^2(\mathrm{Q}))^d.$$

As a result

$$\lim_{h,\delta t\to 0} \int_{\mathrm{Q}} \Lambda_0(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t})(\mathbb{K}\nabla\xi^b_{h,\delta t})\cdot\nabla[\beta(S)]_{h,\delta t}\, \mathrm{d}x\mathrm{d}t = \int_{\mathrm{Q}} \Lambda_0(S;T)(\mathbb{K}\nabla\xi)\cdot\nabla\beta(S)\, \mathrm{d}x\mathrm{d}t.$$

Hence, the fact that $\mathbb{K}$ is symmetric and (6.30) yield

$$\lim_{h,\delta t\to 0} V_2^{h,\delta t} = \lim_{h,\delta t\to 0} W_2^{h,\delta t} = \int_{\mathrm{Q}} \Lambda_0(S;T)\,(\mathbb{K}\nabla\beta(S))\cdot\nabla\xi\;\mathrm{d}x\mathrm{d}t. \tag{6.31}$$

For now, let us establish

$$\lim_{h,\delta t\to 0} W_3^{h,\delta t} = 0.$$

Rearranging the terms of $W_3^{h,\delta t}$, it can be reformulated as follows

$$W_3^{h,\delta t} = \sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}[\Lambda_0]_\tau^{n+1}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}\mathbb{K}_{LM}^\tau(\overline{\gamma}_{LM}^{n+1}-\gamma_{LM}^{n+1}).(S_M^{n+1}-S_L^{n+1}).(\xi_M^n-\xi_L^n),$$

where

$$\overline{\gamma}_{LM}^{n+1} := \begin{cases}\dfrac{\beta(S_M^{n+1})-\beta(S_L^{n+1})}{S_M^{n+1}-S_L^{n+1}} & \text{if } S_L^{n+1}\neq S_M^{n+1},\\ \gamma(S_L^{n+1}) & \text{if } S_L^{n+1}=S_M^{n+1}.\end{cases}$$

Therefore

$$|W_3^{h,\delta t}| \le \Lambda_{0,\max}\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}|\mathbb{K}_{LM}^\tau|.|S_M^{n+1}-S_L^{n+1}|.|\overline{\gamma}_{LM}^{n+1}-\gamma_{LM}^{n+1}|.|\xi_M^n-\xi_L^n|.$$

Using Cauchy–Schwarz inequality, one gets

$$|W_3^{h,\delta t}| \le \Lambda_{0,\max}A_3^{h,\delta t}.(B_3^{h,\delta t})^{\frac{1}{2}}, \tag{6.32}$$

where

$$A_3^{h,\delta t} := \left(\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}\mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}S)^2\right)^2,$$

and

$$B_3^{h,\delta t} := \sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}|\mathbb{K}_{LM}^\tau|.(\overline{\gamma}_{LM}^{n+1}-\gamma_{LM}^{n+1})^2.(\xi_M^n-\xi_L^n)^2.$$

The fact that β^{-1} is θ-Hölder yields

$$|S_M^{n+1}-S_L^{n+1}| \le C_\beta|\beta(S_M^{n+1})-\beta(S_L^{n+1})|^\theta.$$

Therefore, in light of Lemmas 5.3, 5.4 and a priori estimate (5.6), one writes

$$\sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}\mathbb{K}_{LM}^\tau(\delta_{LM}^{n+1}S)^2 \le C_d^2. \tag{6.33}$$

The uniform continuity of γ on the compact set $[0,1]$ yields the existence of a modulus of continuity $\epsilon_2:\mathbb{R}^+\to\mathbb{R}^+$ such that for every $S_1,S_2\in [0,1]$, there holds:

$$|\gamma(S_1)-\gamma(S_2)| \le \epsilon_2(|S_1-S_2|). \tag{6.34}$$

Moreover, ϵ_2 is bounded and

$$\lim_{y\to 0}\epsilon_2(y) = 0.$$

Thus, for every $\tau\in\mathcal{T}$, $\sigma_{LM}^\tau\in\mathcal{E}_\tau$ and $n\in[\![0,l-1]\!]$, one has

$$|\overline{\gamma}_{LM}^{n+1}-\gamma_{LM}^{n+1}| \le \epsilon_2(|\overline{S}_\tau^{n+1}-\underline{S}_\tau^{n+1}|).$$

Therefore

$$B_3^{h,\delta t} \le \sum_{n=0}^{l-1}\delta t\sum_{\tau\in\mathcal{T}}[\epsilon_2(|\overline{S}_\tau^{n+1}-\underline{S}_\tau^{n+1}|)]^2\sum_{\sigma_{LM}^\tau\in\mathcal{E}_\tau}|\mathbb{K}_{LM}^\tau|(\xi_M^n-\xi_L^n)^2.$$

Let us define

$$\epsilon_2^{h,\delta t} := \epsilon_2(|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}|).$$

From Lemma 5.3, one has

$$B_3^{h,\delta t} \leq \int_{\mathrm{Q}} (\epsilon_2^{h,\delta t})^2 (\mathbb{K}\nabla\xi^b_{h,\delta t}) \cdot \nabla\xi^b_{h,\delta t}\,\mathrm{d}x\mathrm{d}t$$
$$\leq K_2 \int_{\mathrm{Q}} (\epsilon_2^{h,\delta t})^2 |\nabla\xi^b_{h,\delta t}|^2\,\mathrm{d}x\mathrm{d}t \leq K_2 \|\nabla\xi^b_{h,\delta t}\|^2_\infty \int_{\mathrm{Q}} (\epsilon_2^{h,\delta t})^2\,\mathrm{d}x\mathrm{d}t. \tag{6.35}$$

Because

$$\lim_{h,\delta t\to 0} \int_{\mathrm{Q}} (\epsilon_2^{h,\delta t})^2\,\mathrm{d}x\mathrm{d}t = 0,$$

one deduces

$$\lim_{h,\delta t\to 0} W_3^{h,\delta t} = \lim_{h,\delta t\to 0} B_3^{h,\delta t} = 0$$

Next, let us prove

$$\lim_{h,\delta t\to 0} W_4^{h,\delta t} = \int_{\mathrm{Q}} \lambda(S;T)\,\eta_w(S;T)\,(\mathbb{K}\nabla P)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t.$$

For that purpose, we rewrite $W_4^{h,\delta t}$ as

$$W_4^{h,\delta t} = \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} \eta_w(;^P)\mathbb{K}^\tau_{LM}\delta^{n+1}_{LM}P.\delta^n_{LM}\xi.$$

Let us set

$$V_4^{h,\delta t} := \int_{\mathrm{Q}} \lambda(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t})\,\eta_w(\underline{S}_{h,\delta t};\underline{T}_{h,\delta t})\,(\mathbb{K}\nabla P_{h,\delta t})\cdot\nabla\xi^b_{h,\delta t}\,\mathrm{d}x\mathrm{d}t.$$

As previously, it can be checked that

$$\lim_{h,\delta t\to 0} V_4^{h,\delta t} = \int_{\mathrm{Q}} \lambda(S;T)\,\eta_w(S;T)\,(\mathbb{K}\nabla P)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t. \tag{6.36}$$

Observe that

$$A_4^{h,\delta t} := |W_4^{h,\delta t} - V_4^{h,\delta t}|$$
$$\leq \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} [\lambda\eta_w]^{n+1}_{LM} |\mathbb{K}^\tau_{LM}||\delta^{n+1}_{LM}P||\delta^n_{LM}\xi|, \tag{6.37}$$

where

$$[\lambda\eta_w]^{n+1}_{LM} := |\lambda_\tau^{n+1}\eta_w(;^P) - \lambda(\underline{S}^{n+1}_\tau;\underline{T}^{n+1}_\tau)\,\eta_w(\underline{S}^{n+1}_\tau;\underline{T}^{n+1}_\tau)|,$$

for every $\tau\in\mathcal{T}$, $\sigma^\tau_{LM}\in\mathcal{E}_\tau$ and $n\in[\![0,l-1]\!]$. Now, the fact that λ and η_w are continuous in S and T yields their uniform continuity on the compact set $[0,1]\times[T_{\min},T_{\max}]$. Hence, there exists two moduli of continuity $\epsilon_3,\epsilon_4:\mathbb{R}^+\to\mathbb{R}^+$ such that for every $S_1,S_2\in[0,1]$ and every $T_1,T_2\in[T_{\min},T_{\max}]$, there holds:

$$|\lambda(S_1,T_1) - \lambda(S_2,T_2)| \leq \epsilon_3(|S_1-S_2| + |T_1-T_2|),$$

and

$$|\eta_w(S_1,T_1) - \eta_w(S_2,T_2)| \leq \epsilon_4(|S_1-S_2| + |T_1-T_2|).$$

Moreover, ϵ_3 and ϵ_4 are bounded and

$$\lim_{y\to 0}\epsilon_3(y) = \lim_{y\to 0}\epsilon_4(y) = 0.$$

Now, by virtue of the triangle inequality, one gets

$$[\lambda\eta_w]^{n+1}_{LM} \leq [\lambda_\tau^{n+1}|\eta_w(;^P) - \eta_w(\underline{S}^{n+1}_\tau;\underline{T}^{n+1}_\tau)| + \eta_w(\underline{S}^{n+1}_\tau;\underline{T}^{n+1}_\tau)|\lambda_\tau^{n+1} - \lambda(\underline{S}^{n+1}_\tau;\underline{T}^{n+1}_\tau)|].$$

Thus

$$[\lambda\eta_w]^{n+1}_{LM} \le [\lambda_2\epsilon_4(|\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau| + |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|) + \epsilon_3(|\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau| + |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|)]$$
$$\le \epsilon_5(|\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau| + |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|), \tag{6.38}$$

where $\epsilon_5 := \lambda_2\epsilon_4 + \epsilon_3 : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded. Therefore

$$\lim_{y\to 0} \epsilon_5(y) = 0.$$

From (6.37) and (6.38), one gets

$$A_4^{h,\delta t} \le \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \epsilon_5(|\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau| + |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|) \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} |\mathbb{K}^\tau_{LM}||\delta^{n+1}_{LM} P||\delta^n_{LM}\xi|.$$

The Cauchy–Schwarz inequality gives

$$A_4^{h,\delta t} \le (B_4^{h,\delta t})^{\frac{1}{2}}.(D_4^{h,\delta t})^{\frac{1}{2}}, \tag{6.39}$$

where

$$B_4^{h,\delta t} := \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} [\epsilon_5(|\overline{S}^{n+1}_\tau - \underline{S}^{n+1}_\tau| + |\overline{T}^{n+1}_\tau - \underline{T}^{n+1}_\tau|)]^2 \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} |\mathbb{K}^\tau_{LM}|(\delta^n_{LM}\xi)^2,$$

and

$$D_4^{h,\delta t} := \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \sum_{\sigma^\tau_{LM}\in\mathcal{E}_\tau} |\mathbb{K}^\tau_{LM}|(\delta^{n+1}_{LM} P)^2.$$

Using the pressure a priori estimate together with Lemma 5.3, one infers

$$D_4^{h,\delta t} \le C_d^3. \tag{6.40}$$

Introducing

$$\epsilon_5^{h,\delta t} := \epsilon_5(|\overline{S}_{h,\delta t} - \underline{S}_{h,\delta t}| + |\overline{T}_{h,\delta t} - \underline{T}_{h,\delta t}|),$$

allows to write

$$B_4^{h,\delta t} \le C_K \int_Q (\epsilon_5^{h,\delta t})^2 (\mathbb{K}\nabla\xi^b_{h,\delta t})\cdot\nabla\xi^b_{h,\delta t}\, \mathrm{d}x\mathrm{d}t$$
$$\le C_d^4 \int_Q (\epsilon_5^{h,\delta t})^2 |\nabla\xi^b_{h,\delta t}|^2\, \mathrm{d}x\mathrm{d}t \le C_d^5 \|\nabla\xi^b_{h,\delta t}\|^2_\infty \int_Q (\epsilon_5^{h,\delta t})^2\, \mathrm{d}x\mathrm{d}t. \tag{6.41}$$

By definition of ϵ_5 and the dominated convergence theorem, there holds

$$\lim_{h,\delta t\to 0} \int_Q (\epsilon_5^{h,\delta t})^2\, \mathrm{d}x\mathrm{d}t = 0. \tag{6.42}$$

Thus, from (6.41), (6.28) and (6.42), one gets

$$\lim_{h,\delta t\to 0} D_4^{h,\delta t} = 0. \tag{6.43}$$

Hence, from (6.39), (6.40) and (6.43), we deduce that

$$\lim_{h,\delta t\to 0} A_4^{h,\delta t} = \lim_{h,\delta t\to 0} |W_4^{h,\delta t} - V_4^{h,\delta t}| = 0.$$

One finally obtains

$$\lim_{h,\delta t\to 0} W_4^{h,\delta t} = \int_Q \lambda(S;T)\,\eta_w(S;T)\,(\mathbb{K}\nabla P)\cdot\nabla\xi\, \mathrm{d}x\mathrm{d}t. \tag{6.44}$$

Similarly, we show that

$$\lim_{h,\delta t\to 0} W_5^{h,\delta t} = \int_Q \lambda(S;T)\,\eta_w(S;T)\,B_o^+(S;T)\,(\mathbb{K}\nabla T)\cdot\nabla\xi\, \mathrm{d}x\mathrm{d}t, \tag{6.45}$$

by considering the function $[\eta_w B_o^+]$ instead of η_w and the temperature gradient $\nabla T_{h,\delta t}$ instead of the pressure gradient $\nabla P_{h,\delta t}$. Indeed, the continuity of the functions η_w and B_o^+ on the compact set $[0,1]\times[T_{\min}, T_{\max}]$ yields the continuity of the function $[\eta_w B_o^+]$ on the same compact set. Thus,

$\eta_w B_o^+$ is uniformly continuous on $[0,1]\times\ [T_{\min}, T_{\max}]$. Therefore, there exists a bounded modulus of continuity associated to $[\eta_w B_o^+]$, and the same reasoning as for $W_4^{h,\delta t}$ is applied to this case.
In the same fashion, by considering $-\,B_o^-$ instead of B_o^+, one finds:

$$\lim_{h,\delta t\to 0} W_6^{h,\delta t} = -\int_{\mathrm{Q}} \lambda(S;T)\,\eta_w(S;T)\,B_o^-(S;T)\,(\mathbb{K}\nabla T)\cdot\nabla\xi \,\mathrm{d}x\mathrm{d}t.$$

This concludes the proof of (6.21). Mimicking the same arguments one can demonstrates (6.22).

(b) Energy conservation equation. Let $\xi\in C_c^\infty(\overline{\Omega}\times[0,t_f))$ be such that $\xi(x,t)=0$ for $(x,t)\in\Gamma_D\times[0,t_f)$. Multiply the 3rd equation of the numerical scheme (4.2) by $\delta t\,\xi_L^n$ where $\xi_L^n=\xi(x_L,t^n)$ for $L\in\ \mathcal{V}\setminus\ \mathcal{V}_D$ and $n\in\ [\![0,l-\ 1]\!]$, then sum over L and n. This yields

$$Y_1^{h,\delta t}+Y_2^{h,\delta t}+Y_3^{h,\delta t}+Y_4^{h,\delta t}+Y_5^{h,\delta t}+Y_6^{h,\delta t}+Y_7^{h,\delta t}+Y_8^{h,\delta t}+Y_9^{h,\delta t}=0, \qquad (6.46)$$

where

$$Y_1^{h,\delta t}=\sum_{n=0}^{l-1}\sum_{L\in\mathcal{V}}|\omega_L|[\psi(\phi_L;S_L^{n+1})T_L^{n+1}-\psi(\phi_L;S_L^n)T_L^n]\xi_L^n,$$

$$Y_2^{h,\delta t}=-\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}[\Lambda_0]_\tau^{n+1}[c_wT](;^S)\mathbb{K}_{LM}^\tau[\beta(S_M^{n+1})-\beta(S_L^{n+1})]\xi_L^n,$$

$$Y_3^{h,\delta t}=-\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}[\Lambda_0]_\tau^{n+1}[c_wT](;^S)\mathbb{K}_{LM}^\tau\{\gamma_{LM}^{n+1}(S_M^{n+1}-S_L^{n+1})-[\beta(S_M^{n+1})-\beta(S_L^{n+1})]\}\xi_L^n,$$

$$Y_4^{h,\delta t}\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}[\Lambda_0]_\tau^{n+1}[c_oT](;^S)\mathbb{K}_{LM}^\tau[\beta(S_M^{n+1})-\beta(S_L^{n+1})]\xi_L^n,$$

$$Y_5^{h,\delta t}=\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}[\Lambda_0]_\tau^{n+1}[c_oT](;^S)\mathbb{K}_{LM}^\tau\{\gamma_{LM}^{n+1}(S_M^{n+1}-S_L^{n+1})-[\beta(S_M^{n+1})-\beta(S_L^{n+1})]\}\xi_L^n,$$

$$Y_6^{h,\delta t}=-\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}[\Lambda_0]_\tau^{n+1}[c_oT]\lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)T](;^P)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}P\xi_L^n,$$

$$Y_7^{h,\delta t}=-\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}\lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^+\,T](;^T)\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T\xi_L^n,$$

$$Y_8^{h,\delta t}=\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}\lambda_\tau^{n+1}[(c_w\eta_w+c_o\eta_o)B_o^-\,T](;^{-T})\mathbb{K}_{LM}^\tau\delta_{LM}^{n+1}T\xi_L^n,$$

$$Y_9^{h,\delta t}=\sum_{n=0}^{l-1}\delta t\sum_{L\in\mathcal{V}}\sum_{\substack{\tau\in\tau_L\\ M\in\mathcal{V}_\tau\setminus\{L\}}}(\kappa_T)_{LM}^\tau\delta_{LM}^{n+1}T\xi_L^n.$$

The accumulation term is handled as done for $W_1^{h,\delta t}$ in the 1st equation. After its processing, it can be written under the form

$$Y_1^{h,\delta t}=-\int_{\mathrm{Q}}\psi(\widetilde{\phi}_h,\widetilde{S}_{h,\delta t})\widetilde{T}_{h,\delta t}\widetilde{\zeta}_h^1(x,t)\,\mathrm{d}x\mathrm{d}t-\int_\Omega\psi(\widetilde{\phi}_h,\widetilde{S}_h^0)\widetilde{T}_h^0\widetilde{\xi}_h^0\,\mathrm{d}x,$$

where

$$\begin{aligned}
\widetilde{\zeta}_h^1(x,t) &= \partial_t \xi(x_L,t) && \text{for } x \in \omega_L \text{ and } t \in [0,t_f),\\
\widetilde{S}_h^0(x) &= S_L^0 && \text{for } x \in \omega_L,\\
\widetilde{T}_h^0(x) &:= T_L^0 && \text{for } x \in \omega_L,\\
\widetilde{\xi}_h^0(x) &= \xi(x_L,0) && \text{for } x \in \omega_L.
\end{aligned}$$

Because $\xi \in C_c^\infty(\Omega \times [0,t_f))$, $S^0 \in L^\infty(\Omega)$ and $T^0 \in L^\infty(\Omega)$, one gets

$$\widetilde{\zeta}_h^1 \longrightarrow \partial_t \xi, \quad \widetilde{S}_h^0 \longrightarrow S^0, \quad \widetilde{T}_h^0 \longrightarrow T^0, \quad \widetilde{\xi}_h^0 \longrightarrow \xi(\,\cdot\,,0) \quad \text{a.e. on } \Omega.$$

From Proposition 6.3, it follows that

$$\widetilde{\phi}_h \longrightarrow \phi \quad \text{a.e. on } \Omega, \qquad \widetilde{S}_{h,\delta t} \longrightarrow S \quad \text{a.e. on Q}, \qquad \widetilde{T}_{h,\delta t} \longrightarrow T \quad \text{a.e. on Q}.$$

The dominated convergence theorem together with the fact that all involved sequences of functions are bounded yield

$$\lim_{h,\delta t\to 0} Y_1^{h,\delta t} = -\int_{\mathrm{Q}} \psi(\phi,S) T \partial_t \xi \,\mathrm{d}x\mathrm{d}t - \int_\Omega \psi(\phi,S^0) T^0 \xi(x,0)\,\mathrm{d}x. \tag{6.47}$$

Similarly to $W_2^{h,\delta t}$ in the 1st equation, one shows that

$$\lim_{h,\delta t\to 0} Y_2^{h,\delta t} = \int_{\mathrm{Q}} \Lambda_0(S;T)\, c_w\, T\, (\mathbb{K}\nabla\beta(S)) \cdot \nabla\xi \,\mathrm{d}x\mathrm{d}t. \tag{6.48}$$

For this purpose, one considers Λ_0 instead of λ, $[c_w T]$ instead of η_w and the capillary term gradient $\nabla[\beta(S)]_{h,\delta t}$ instead of the pressure gradient $\nabla P_{h,\delta t}$. Observe that $[c_w T]$ is Lipschitz continuous in T. Therefore, there exists a bounded modulus of continuity associated to $[c_w T]$, and the same reasoning as for $W_4^{h,\delta t}$ is also applied to this case.

Bearing in mind the temperature maximum principle as well as the fact that the functions $[c_w T]$, $[c_o T]$, B_o^+, B_o^- are bounded. Using analogous arguments as in the proof of $W_3^{h,\delta t}, W_4^{h,\delta t}$ in the 1st equation, it can be easily seen that

$$\lim_{h,\delta t\to 0} Y_3^{h,\delta t} = 0, \tag{6.49}$$

$$\lim_{h,\delta t\to 0} Y_4^{h,\delta t} = -\int_{\mathrm{Q}} \Lambda_0(S;T)\, c_o\, T\, (\mathbb{K}\nabla\beta(S)) \cdot \nabla\xi \,\mathrm{d}x\mathrm{d}t, \tag{6.50}$$

$$\lim_{h,\delta t\to 0} Y_5^{h,\delta t} = 0. \tag{6.51}$$

$$\lim_{h,\delta t\to 0} Y_6^{h,\delta t} = \int_{\mathrm{Q}} [\lambda(c_w\eta_w \,+\, c_o\eta_o)](S;T) T (\mathbb{K}\nabla P) \cdot \nabla\xi \,\mathrm{d}x\mathrm{d}t, \tag{6.52}$$

$$\lim_{h,\delta t\to 0} Y_7^{h,\delta t} = \int_{\mathrm{Q}} [\lambda(c_w\eta_w \,+\, c_o\eta_o)B_o^+](S;T) T (\mathbb{K}\nabla T) \cdot \nabla\xi \,\mathrm{d}x\mathrm{d}t, \tag{6.53}$$

$$\lim_{h,\delta t\to 0} Y_8^{h,\delta t} = -\int_{\mathrm{Q}} [\lambda(c_w\eta_w \,+\, c_o\eta_o)B_o^-](S;T) T (\mathbb{K}\nabla T) \cdot \nabla\xi \,\mathrm{d}x\mathrm{d}t. \tag{6.54}$$

Finally, regarding the term $Y_9^{h,\delta t}$, one writes

$$\begin{aligned}
Y_9^{h,\delta t} &= \sum_{n=0}^{l-1} \delta t \sum_{L\in\mathcal{V}} \sum_{\substack{\tau\in\tau_L \\ M\in\mathcal{V}_\tau\setminus\{L\}}} [\Lambda_0]_\tau^{n+1} [c_o T] (\kappa_T)_{LM}^\tau \delta_{LM}^{n+1} T \xi_L^n \\
&= \sum_{n=0}^{l-1} \delta t \sum_{\tau\in\mathcal{T}} \lambda_\tau^{n+1} \sum_{\sigma_{LM}^\tau \in\mathcal{E}_\tau} (\kappa_T)_{LM}^\tau \delta_{LM}^{n+1} T \cdot \delta_{LM}^n \xi = \int_{\mathrm{Q}} (\kappa_T \nabla\xi_{h,\delta t}^b) \cdot \nabla T_{h,\delta t} \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{6.55}$$

Now, note that $\kappa_T \in (L^\infty(\Omega))^{d\times d}$ yields

$$\kappa_T \nabla \xi^b_{h,\delta t} \longrightarrow \kappa_T \nabla \xi \quad \text{strongly in } (L^2(\mathrm{Q}))^d.$$

Moreover, one has

$$\nabla T_{h,\delta t} \longrightarrow \nabla T \quad \text{weakly in } (L^2(\mathrm{Q}))^d.$$

Thus

$$\lim_{h,\delta t\to 0} Y_9^{h,\delta t} = \lim_{h,\delta t\to 0} \int_{\mathrm{Q}} (\kappa_T \nabla \xi^b_{h,\delta t}) \cdot \nabla T_{h,\delta t} \, \mathrm{d}x\mathrm{d}t = \int_{\mathrm{Q}} (\kappa_T \nabla \xi) \cdot \nabla T \, \mathrm{d}x\mathrm{d}t. \tag{6.56}$$

Thus, by gathering all the limits (6.47), (6.48)–(6.56) in formula (6.46), one proves (6.23). This concludes the proof of the passage to the limit theorem. ■

7. Numerical results

In this section, we present the numerical results for 2D test cases modeling different scenarios of nonisothermal immiscible incompressible two-phase flow in porous media using the numerical scheme studied above. The first test case is a simulation in a homogeneous reservoir while the second one is in a heterogeneous domain. These test cases are presented to evaluate the efficiency, robustness and accuracy of the developed finite volume scheme. All our developments have been implemented in DuMu^X [25]. It provides many tools to solve numerically PDEs and allowing, among other things, the management of mesh, discretization or linear and nonlinear solvers. The code is an object-oriented software written in C++ and has massively parallel computation capability. The modular concept of DuMu^X makes it easy to integrate new modules adapted to our numerical scheme. More precisely, we have implemented a new module that utilizes the above-described scheme and the BBOX module of DuMu^X for spatial discretization. The simulation were performed on a laptop with Intel(R) Core(TM) i7-4810MQ CPU Processor 2.80 GHz with 16 GB RAM.

7.1. Case 1: Homogeneous porous medium

In this test, we are interested in the accuracy evaluation of the proposed finite volume scheme, for a nonisothermal two-phase flow through a 2D homogeneous porous medium. The purpose of this test case is to show the numerical convergence in the L^2 norm of the numerical scheme. This test case was adapted from a numerical test in [28] modeling CO_2 injection into a layered aquifer. We consider an immiscible incompressible two-phase flow model instead of two-phase two-component flow.

The incompressible nonwetting phase is injected into a 2D rectangular aquifer. The domain is a rectangle of length $200\,m$ and height $100\,m$ with a depth of the bottom boundary of $1200\,m$. The porous medium is a homogeneous reservoir with porosity $\phi = 0.2$ and absolute permeability $K = 3\times10^{-14}[m^2]$. The phase densities are $\rho_w = 1000\,kg.m^{-3}$, $\rho_o = 635\,kg.m^{-3}$ and $\rho_s = 2700\,kg.m^{-3}$. The specific heat capacity of the solid matrix is $C_s = 790\,J.K^{-1}.kg^{-1}$. The Brooks–Corey [24] model is considered for the capillary pressure and the relative permeabilities functions with the parameters $\nu = 2$, the entry pressure $P_e = 10^4\,Pa$ and the residual saturations $S_{wr} = S_{or} = 0$. Therefore, $k_{rw}(S_w) = S_w^4$, $k_{ro}(S_w) = (1 - S_w)^2(1 - S_w^2)$ and $P_c(S_w) = P_e\,S_w^{-\frac{1}{2}}$. The Somerton model [31] is considered for the thermal conductivity. Due to the complexity of the model, we refer the reader to [28] and the references therein, where all remaining parameters such as phase viscosities and thermal conductivity, are provided.

The domain is initially fully saturated with brine H_2O liquid. We assume hydrostatic pressure and temperature at the start of the simulation, with temperature gradient $0.03K.m^{-1}$. The nonwetting phase is injected over a period of 3.17 years from the left boundary over a height of $30\,m$ into the aquifer with a rate of $0.003\, kg.s^{-1}$ which yields the energy flux equal to $-10^{-4}\, h_o(T_{inj}, P_{inj})$, where h_o stands for the nonwetting enthalpy and is a function of the injection temperature $T_{inj} = 305K$ and pressure $P_{inj} = 16\, 10^6\, Pa$. On the right boundary, we impose Dirichlet conditions in accordance with the initial conditions. Elsewhere, a homogeneous Neumann boundary conditions is imposed.

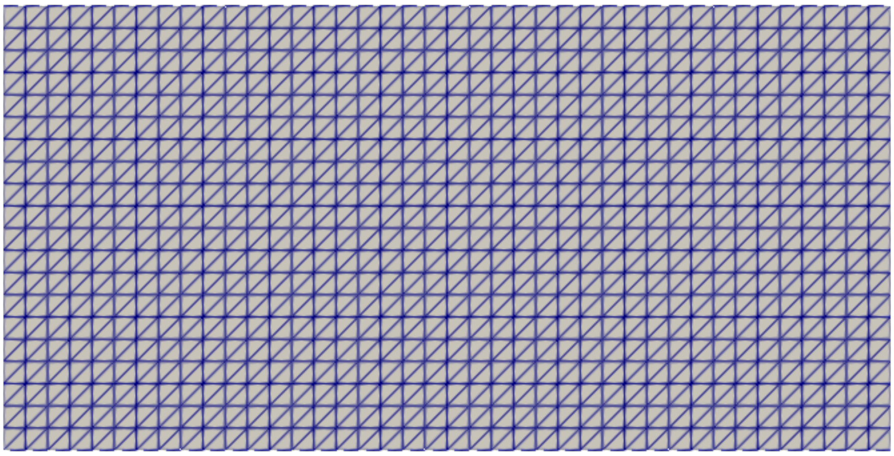

Figure 7.1. Triangular mesh of Ω with 1600 triangles.

The numerical test was implemented with several meshes, including the most refined of structured triangular mesh with 102400 triangles. The type of the used mesh is illustrated in Figure 7.1.

The nonlinear system is solved by the Newton method and a BiConjugate Gradient STABilized (BiCGSTAB) method, preconditioned by an Algebraic Multigrid (AMG) solver, is used to solve the linear systems. The tolerances for the Newton and the BICGSTAB methods are respectively 10^{-8} and 10^{-13}. The simulation starts with an initial time step $\delta t^0 = 10^3\, s$ and a maximal time step $\delta t_{max} = 10^5\, s$ is imposed. A minimal time step of $\delta t_{min} = 10^3\, s$ was registered. Time step sizes during transient simulations are dynamically recalculated depending on the convergence behavior of the Newton method which can be increased or reduced, depending on the number of iterations allowed in each nonlinear iteration. In this case, Newton's method converges rapidly in less than 5 iterations. The number of iterations before reaching the maximal time step is 8.

We can say that all quantities of interest: the nonwetting phase pressure (P_o), saturation (S_o) and the temperature T behave as expected without instabilities. The results of these simulations are omitted since nothing startling was found. Instead, we concentrate to provide a quantitative study for the numerical convergence of the proposed finite volume scheme. For that, we compute the L^2 relative error on the nonwetting saturation and pressure, and the temperature on different structured triangular meshes, with 1600, 6400 and 25600 triangles. We have considered the solution of the previous simulation on a structured triangular mesh, with 102400 triangles, as a reference solution.

In order to clarify things, let's begin by defining the L^2 relative errors when utilizing the finest mesh as a reference solution. For instance, if we consider the temperature variable, let $\mathcal{V}$ be the set of vertices of the coarse mesh, $(\omega_L)_{L\in\mathcal{V}}$ the set of control volumes of the coarse mesh, T^l the approximate solution on the coarse mesh, and $T^{ref,l}$ the approximate reference solution on the finest mesh at the final time t^l. In this case, the L^2 relative error is computed at the final time $t^l = 3.17$ years as follows:

$$\textbf{Relative\_Error} = \frac{(\sum_{L\in\mathcal{V}} |\omega_L|(T_L^l - T_L^{ref,l})^2)^{\frac{1}{2}}}{(\sum_{L\in\mathcal{V}} |\omega_L|(T_L^{ref,l})^2)^{\frac{1}{2}}}.$$

Table 7.1. L^2 relative errors on the nonwetting saturation, pressure, and the temperature for 3 structured triangular meshes.

Refinement level	Number of triangles	NDOF	S_o	P_o	T	CPU time
1	1600	861	2.302e-01	1.337e-03	3.788e-04	2 mn
2	6400	3321	1.55e-01	7.817e-04	1.753e-04	16 mn
3	25600	13041	7.987e-02	3.334e-04	6.307e-05	1 h 25 mn
4	102400	51681	-	-	-	8 h 45 mn

Indeed, the solution on the coarse mesh is defined on the vertices. These same degrees of freedom are also recovered from the refined mesh. Then, the results are compared to compute the error. Consequently, the L^2 relative errors for the 3 considered meshes are reported in Table 7.1. The associated curves are represented in logarithmic scale (for both axes) in Figure 7.2. We can clearly see that the errors diminish with each refinement step, the errors for the temperature and the nonwetting pressure being smaller than that of the saturation. Table 7.1 also gives the CPU times for all 4 simulations.

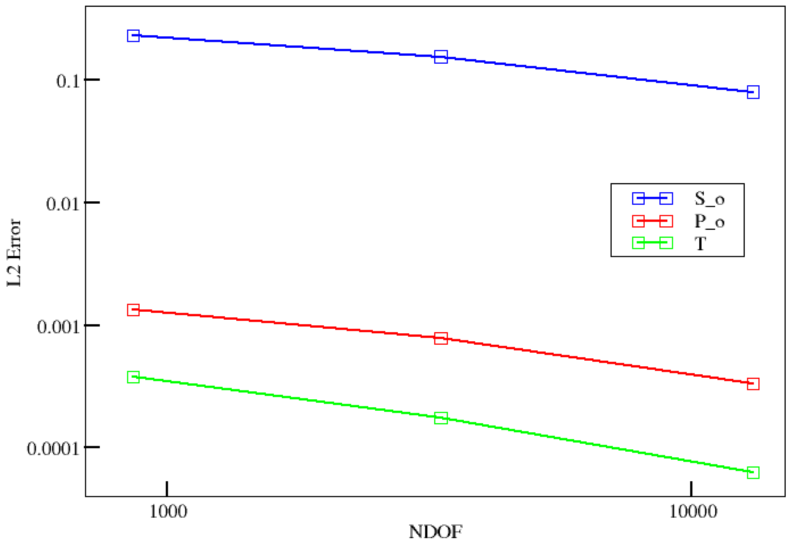

Figure 7.2. L^2 relative error in logarithmic scale (for both axes) for the temperature, the nonwetting saturation and pressure.

To complete our convergence analysis, we assessed the convergence rates for the 3 quantities of interest as Table 7.2 highlights.

Table 7.2. Orders of convergence for the L^2 relative errors.

	S_o	P_o	T
Order of convergence	0.76356	1.0018	1.2933

We can see that the temperature has the greatest order of convergence which is in agreement with the diffusive effect of the temperature. Indeed, the temperature gradient appears in the elliptic term associated to the thermal diffusion of the energy equation. Furthermore, we see that the order of convergence for the pressure is slightly greater than 1 which is in accordance with the convective effect of the pressure gradient and thus the regularity of the pressure. Lastly, the saturation has the smallest order of convergence, much bellow 1. This can be explained by the discontinuity of the saturation which manifests as sharp fronts.

7.2. **Case 2: Heterogeneous porous medium**

The purpose of this test case is to show the performance of the finite volume scheme on 2D heterogeneous porous media with curved geometry. We present the numerical results for a test case adapted from [28] modeling CO2 injection into a layered aquifer. We consider the same data as in the first test case, in a heterogeneous porous medium. The incompressible nonwetting phase is injected, over a period of 6.34 years, into a 2D rectangular aquifer. We are looking at a simplified geological configuration that represents a 2D vertical section of a reservoir. The domain is a rectangle of length $200\,m$ and height $100\,m$ with a depth of the bottom boundary of $1200\,m$. It is composed of four sub-domains of different permeabilities and porosities as shown on the left in Figure 7.3. Apart from that, all remaining data is that of the first test case. The considered mesh is comprised of 12987 triangles and 6598 vertices (see Figure 7.3 on the right). The triangles aspect ratios vary from 1.001 to 2.031, and their areas vary from $0.161\,m^2$ to $24.44\,m^2$.

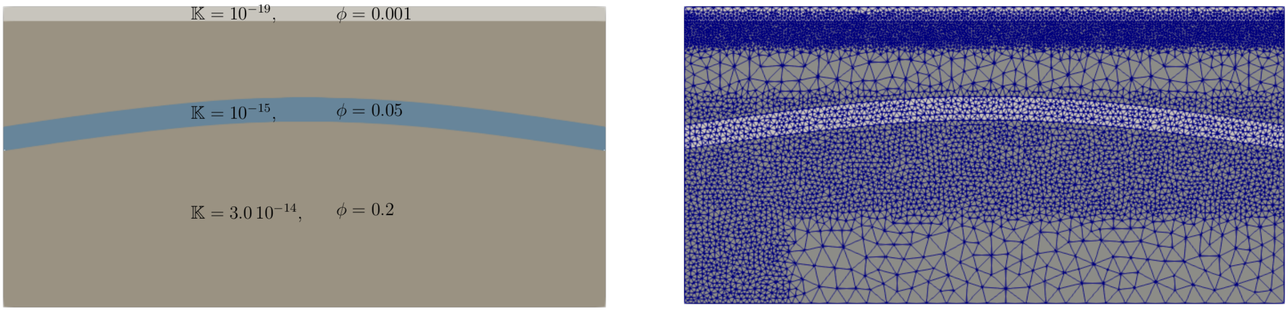

Figure 7.3. Permeability and porosity fields (left). Unstructured grids used for the simulation (right).

With regards to simulation parameters, the tolerances for the Newton and the BICGSTAB methods are respectively 10^{-8} and 10^{-13}. The simulation starts with an initial time step $\delta t^0 = 10^3\,s$ and a maximal time step $\delta t_{max} = 10^5\,s$ is imposed. A minimal time step of $\delta t_{min} = 10^3\,s$ was registered. In this case, Newton's method converges rapidly in less than 5 iterations. Let us underline that the theoretical convergence of the Newton solver is difficult to address in two-phase flows in porous media. In the case of Richards equation, the authors of the work [27] were able to prove the Newton solver convergence.

The numerical results are shown in Figures 7.4–7.5 below. The represented quantities are: the pressure (P_o), the saturation (S_o), the overpressure, and the temperature T. The quantity injected rises rapidly to the top due to the large difference in densities and there is influenced by the permeability variations in the field. The front is very sharp due to the strong and sharp localized variation of the permeability which is remarkably captured. The nonwetting phase remains trapped at the top of the third layer due to the very low permeability of the fourth layer. It is worth remarking that saturation scale ranges from 0 to 1 and no over/undershooting occurs.

The overpressure is defined as the difference between the current and initial nonwetting pressure. Positive values show that we are always in overpressure during the whole simulation. The wetting pressure curves show that the pressure increases rapidly with an overpressure reaching $3.5\,10^5 Pa$ at $t = 1.59$ years and returning at the end of the simulation to an overpressure around $1.9\,10^5 Pa$. The

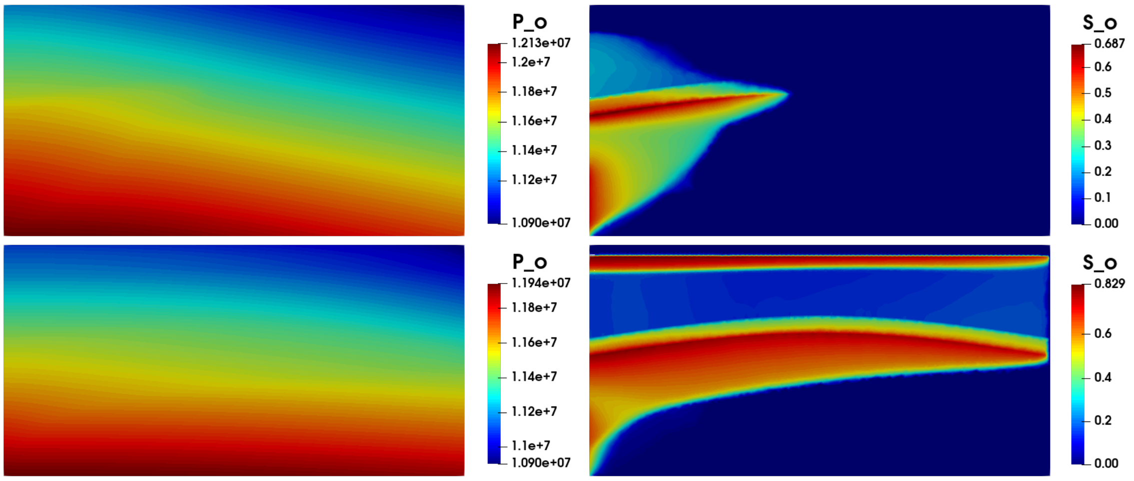

FIGURE 7.4. Profiles of the nonwetting pressure (left) and saturation (right) at 2 different times. From top to bottom: $t_1 = 1.59$ years and the final time $t_f = 6.34$ years.

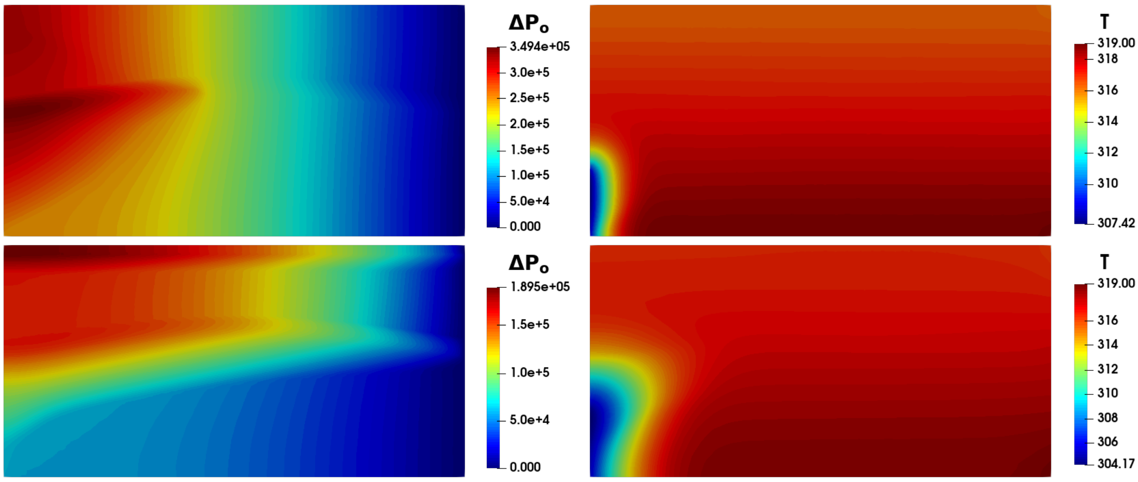

FIGURE 7.5. Profiles of the overpressure (left) and the temperature (right) at 2 different times. From top to bottom: $t_1 = 1.59$ years and the final time $t_f = 6.34$ years.

injection zone can cool and maintain a constant pressure in this zone because the temperature of the injection zone is lower than the reservoir temperature. Heat transfer seems to be driven more by conduction than convection, and it is only in the near injection region that the temperature is modified and respects the discrete maximum principle. We achieved a good agreement between our results and those of [28].

8. Concluding remarks

We have presented a vertex-centered finite volume scheme for solving a system of coupled degenerate PDEs modeling nonisothermal incompressible immiscible two-phase in heterogeneous porous media. The concept of the global pressure, developed specifically for nonisothermal flows, is used to demonstrate the convergence of the numerical approximation to a weak solution based on a priori estimates and compactness arguments. The method was validated through numerical results on a scenario of geological storage of CO_2. This work is an extension of the results established in [21] in the isothermal case to the problem of nonisothermal incompressible two-phase flow in heterogeneous porous media. To our best knowledge, this is the first convergence result for a CVFE scheme in the case of non-isothermal two-phase flow in heterogeneous porous media. It's worth noting that a convergence study was performed recently in [5] on the discretization of a system that models nonisothermal compressible two-phase flow in porous media using a cell-centered TPFA finite volume method. However, the study still needs to be improved in several areas, including discontinuous and degenerate capillary pressures, media that is highly heterogeneous, fractured, or anisotropic and by developing a general approach to incorporating compressibility of both phases. Finally, in this paper, the validation was carried out on 2D cases; it would be interesting to have a more detailed validation for 3D cases. These more complex cases are being displayed in the applications. These crucial issues require more in-depth work.

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References

[1] L. Agélas and R. Masson. Convergence of the finite volume MPFA O scheme for heterogeneous anisotropic diffusion problems on general meshes. *C. R. Math. Acad. Sci. Paris*, 346(17-18):1007–1012, 2008.

[2] L. Agélas, M. Schneider, G. Enchéry, and B. Flemisch. Convergence of nonlinear finite volume schemes for two-phase porous media flow on general meshes. *IMA J. Numer. Anal.*, 42(1):515–568, 2022.

[3] E. Ahusborde, B. Amaziane, F. Croccolo, and N. Pillardou. Numerical simulation of a thermal-hydraulic-chemical multiphase flow model for CO2 sequestration in saline aquifers. *Math. Geosci.*, 56:541–572, 2024.

[4] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183:311–341, 1983.

[5] B. Amaziane, M. El Ossmani, and Y. Zahraoui. Convergence of a TPFA finite volume scheme for non-isothermal immiscible compressible two-phase flow in porous medias. *Comput. Math. Appl.*, 165:118–149, 2024.

[6] B. Amaziane, M. Jurak, L. Pankratov, and A. Piatnitski. An existence result for nonisothermal immiscible incompressible 2-phase flow in heterogeneous porous media. *Math. Methods Appl. Sci.*, 40(18):7510–7539, 2017.

[7] B. Amaziane, M. Jurak, and I. Radišić. Convergence of a finite volume scheme for immiscible compressible two-phase flow in porous media by the concept of the global pressure. *J. Comput. Appl. Math.*, 399: article no. 113728 (26 pages), 2022.

[8] B. Andreianov, M. Bendahmane, and R. Ruiz-Baier. Analysis of a finite volume method for a cross-diffusion model in population dynamics. *Math. Models Methods Appl. Sci.*, 21(2):307–344, 2011.

[9] G. R. Barrenechea, V. John, and P. Knobloch. Finite element methods respecting the discrete maximum principle for convection-diffusion equations. *SIAM Rev.*, 66(1):3–88, 2024.

[10] L. Beaude, R. Masson, S. Lopez, and P. Samier. Combined face based and nodal based discretizations on hybrid meshes for non-isothermal two-phase Darcy flow problems. *ESAIM, Math. Model. Numer. Anal.*, 53(4):1125–1156, 2019.

[11] M. Beneš. Analysis of non-isothermal multiphase flows in porous media. *Math. Methods Appl. Sci.*, 45(16):9653–9677, 2022.

[12] M. Beneš. Weak solutions of coupled variable-density flows and heat transfer in porous media. *Nonlinear Anal., Theory Methods Appl.*, 221: article no. 112973 (27 pages), 2022.

[13] O. B. Bocharov and V. N. Monakhov. Boundary value problems of nonisothermal two-phase filtration in porous media. In *Free boundary problems in fluid flow with applications (Montreal, PQ, 1990)*, volume 282 of *Pitman Research Notes in Mathematics Series*, pages 166–178. Longman Scientific & Technical, 1993.

[14] K. Brenner, R. Masson, E. H. Quenjel, and J. Droniou. Total velocity-based finite volume discretization of two-phase Darcy flow in highly heterogeneous media with discontinuous capillary pressure. *IMA J. Numer. Anal.*, 42(2):1231–1272, 2022.

[15] C. Cancès, J. Droniou, C. Guichard, G. Manzini, M. B. Olivares, and I. S. Pop. Error estimates for the gradient discretisation method on degenerate parabolic equations of porous medium type. In *Di Pietro, D. A., Formaggia, L., Masson, R. (eds) Polyhedral Methods in Geosciences*, volume 27 of *SEMA SIMAI Springer Series*, pages 37–72. Springer, 2021.

[16] C. Cancès, I. S. Pop, and M. Vohralík. An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow. *Math. Comput.*, 83(285):153–188, 2014.

[17] X. Cao, S. F. Nemadjieu, and I. S. Pop. Convergence of an MPFA finite volume scheme for a two-phase porous media flow model with dynamic capillarity. *IMA J. Numer. Anal.*, 39(1):512–544, 2019.

[18] J. Droniou. Finite volume schemes for diffusion equations: Introduction to and review of modern methods. *Math. Models Methods Appl. Sci.*, 24(8):1575–1619, 2014.

[19] L. C. Evans. *Partial Differential Equations, Second edition*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.

[20] B. Faigle, M. A. Elfeel, R. Helmig, B. Becker, B. Flemisch, and S. Geiger. Multi-physics modeling of non-isothermal compositional flow on adaptive grids. *Comput. Methods Appl. Mech. Eng.*, 292:16–34, 2015.

[21] M. Ghilani, M. Saad, and E. H. Quenjel. Positive control volume finite element scheme for degenerate compressible two-phase flow in anisotropic porous media. *Comput. Geosci.*, 23(1):55–79, 2019.

[22] M. Ghilani, M. Saad, and E. H. Quenjel. Positivity-preserving finite volume scheme for compressible two-phase flows in anisotropic porous media: the densities are depending on the physical pressures. *J. Comput. Phys.*, 407: article no. 109233 (29 pages), 2020.

[23] D. Gläser, B. Flemisch, R. Helmig, and H. Class. A hybrid-dimensional discrete fracture model for non-isothermal two-phase flow in fractured porous media. *GEM Int. J. Geomath.*, 10(1): article no. 5 (25 pages), 2019.

[24] R. Helmig. *Multiphase Flow and Transport Processes in the Subsurface: A Contribution to the Modeling of Hydrosystems.* Springer, 1997.

[25] T. Koch, D. Gläser, K. Weishaupt, S. Ackermann, M. Beck, B. Becker, S. Burbulla, H. Class, E. Coltman, and S. Emmert. DuMux 3 – an open-source simulator for solving flow and transport problems in porous media with a focus on model coupling. *Comput. Math. Appl.*, 81(1):423–443, 2021.

[26] S. Korotov and M. Křížek. Nonobtuse local tetrahedral refinements towards a polygonalface/interface. *Appl. Math. Lett.*, 24(6):817–821, 2011.

[27] F. A. Radu, I. S. Pop, and P. Knabner. Newton-type methods for the mixed finite element discretization of some degenerate parabolic equations. In *Numerical Mathematics and Advanced Applications*, pages 1192–1200. Springer, 2006.

[28] M. Schneider, B. Flemisch, and R. Helmig. Monotone nonlinear finite-volume method for nonisothermal two-phase two-component flow in porous media. *Int. J. Numer. Methods Fluids*, 84(6):352–381, 2017.

[29] M. Schneider, B. Flemisch, R. Helmig, K. Terekhov, and H. Tchelepi. Monotone nonlinear finite-volume method for challenging grids. *Comput. Geosci.*, 22(2):565–586, 2018.

[30] C. J. Seeton. Viscosity-temperature correlation for liquids. *Tribol. Lett.*, 22(1):67–78, 2006.

[31] W. H. Somerton, J. Keese, and S. L. Chu. Thermal behavior of unconsolidated oil sands. *SPE J.*, 14:513–521, 1974.