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A statistical approach for simulating the density solution of a McKean–Vlasov equation

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Abstract. We prove optimal convergence results of a stochastic particle method for computing the classical solution of a multivariate McKean–Vlasov equation, when the measure variable is in the drift, following the classical approach of [1, 11]. Our method builds upon adaptive nonparametric results in statistics that enable us to obtain a data-driven selection of the smoothing parameter in a kernel type estimator. In particular, we generalise the Bernstein inequality of [18] for mean-field McKean–Vlasov models to interacting particles Euler schemes and obtain sharp deviation inequalities for the estimated classical solution. We complete our theoretical results with a systematic numerical study, and gather empirical evidence of the benefit of using high-order kernels and data-driven smoothing parameters.

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1. Introduction

1.1. Motivation

Let \mathcal{P}_1 denote the set of probability distributions on \mathbb{R}^d , $d \geq 1$, with at least one moment. Given a time horizon $T > 0$, an initial condition $\mu_0 \in \mathcal{P}_1$ and two functions

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}^d \quad \text{and} \quad \sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d,$$

a probability solution $\mu = (\mu_t(dx))_{t \in [0, T]}$ of the nonlinear Fokker–Planck equation

$$\begin{cases} \partial_t \mu + \operatorname{div}(b(\cdot, \cdot, \mu)\mu) = \frac{1}{2} \sum_{k, k'=1}^d \partial_{kk'}^2 (\sigma \sigma^\top \mu) \\ \mu_{t=0} = \mu_0 \end{cases} \quad (1.1)$$

exists, under suitable regularity assumptions on the data (μ_0, b, σ) , see in particular Assumptions A1, A2 and (A3 or A4) in Section 2.1 and 2.4 below. Moreover, for each $t > 0$, $\mu_t(dx)$ has a smooth density, also denoted by $\mu_t(x)$, which is a classical solution to (1.1). There is a vast literature on approximating (1.1), we mention [19, 31, 35, 48, 52] to name but a few. The objective of the paper is to construct a probabilistic numerical method to compute $\mu_T(x)$ for every $T > 0$ and $x \in \mathbb{R}^d$, with accurate statistical guarantees, that improves on previous works on the topic using an interacting particle system simulated from an Euler scheme. The main novelty is the introduction of an adaptive nonparametric statistical methodology, that enables us to obtain confidence intervals for the classical

solution of (1.1), and this departs from what is usually sought when controlling the classical weak or strong error in an Euler scheme.

Model (1.1) includes in particular nonlinear transport terms of the form $b(t, x, \mu) = F(f(x) + \int_{\mathbb{R}^d} g(x-y)\mu(dy))$, for smooth functions $F, f, g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ that account for evolution systems with common force f and mean-field interaction g under some nonlinear transformation F together with diffusion σ . Such models and their generalizations have inspired a myriad of application domains over the last decades, ranging from physics [10, 41] to neurosciences [2, 9], structured models in population dynamics (like in e.g. swarming and flocking models [2, 9, 13, 44]), together with finance and mean-field games [14, 15, 34], social sciences and opinion dynamics [16], to cite but a few.

The classical approach of Bossy and Talay [11], subsequently improved by Antonelli and Kohatsu-Higa [1] paved the way: the probability μ_t has a natural interpretation as the distribution $\mathcal{L}(X_t)$ of an \mathbb{R}^d -valued random variable X_t , where the stochastic process $(X_t)_{t \in [0, T]}$ solves a McKean–Vlasov equation

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t)dB_t \\ \mathcal{L}(X_0) = \mu_0, \end{cases} \quad (1.2)$$

for some standard d -dimensional Brownian motion $B = (B_t)_{t \in [0, T]}$. The mean-field interpretation of (1.2) is the limit as $N \rightarrow \infty$ of an interacting particle system

$$\begin{cases} dX_t^n = b(t, X_t^n, N^{-1} \sum_{m=1}^N \delta_{X_t^m})dt + \sigma(t, X_t^n)dB_t^n, \quad 1 \leq n \leq N, \\ \mathcal{L}(X_0^1, \dots, X_0^N) = \mu_0^{\otimes N}, \end{cases}$$

where the $(B_t^n)_{t \in [0, T]}$ are independent Brownian motions, and δ_a denotes the Dirac measure at a point $a \in \mathbb{R}^d$. Indeed, under fairly general assumptions, see e.g. [42, 43, 51] we have $N^{-1} \sum_{n=1}^N \delta_{X_t^n} \rightarrow \mu_t$ weakly as $N \rightarrow \infty$. One then simulates an approximation of

$$(X_t^1, \dots, X_t^N)_{t \in [0, T]} \quad (1.3)$$

by a stochastic system of particles

$$(\bar{X}_t^{1, h}, \dots, \bar{X}_t^{N, h})_{t \in [0, T]} \quad (1.4)$$

using an Euler scheme with mesh h , see (2.2) below for a rigorous definition. The synthesised data (1.4) are then considered as a proxy of the system (1.3) and a simulation of $\mu_T(x)$ is obtained via the following nonparametric kernel estimator

$$\hat{\mu}_T^{N, h, \eta}(x) := N^{-1} \sum_{n=1}^N \eta^{-d} K(\eta^{-1}(x - \bar{X}_T^{n, h})). \quad (1.5)$$

Here $K : \mathbb{R}^d \rightarrow \mathbb{R}$ is kernel function that satisfies $\int_{\mathbb{R}^d} K(x)dx = 1$ together with additional moment and localisation conditions, see Definition 2.1 below, and η is the statistical smoothing parameter. Note that the computational cost of (1.5) with a tensor-product kernel and coordinate-wise bandwidths (see also Definition 2.1 and Remark 2.2) at one point x is of order $O(Nd)$, where N is the number of particles and d the dimension, which is the same as in standard kernel density estimation with i.i.d. samples.

When K is a Gaussian kernel, [11] and [1] obtained some convergence rates that depend on N, η, h and the data (μ_0, b, σ) . See also the comprehensive review of Bossy [8]. Our objective is somehow to simplify and improve on these results, up to some limitations of course, by relying on adaptive statistical methods. In order to control the strong error $\mathbb{E} [|\hat{\mu}_T^{N, h, \eta}(x) - \mu_T(x)|^2]$, a crucial step is to select the optimal bandwidth η that exactly balances the bias and the variance of the statistical error and that depends on the smoothness of the map $x \mapsto \mu_T(x)$. To do so, we rely on recent data-driven selection procedures for η based on the Goldenshluger–Lepski’s method in statistics [23, 24, 25].

They do not need any prior knowledge of $x \mapsto \mu_T(x)$, and this is a major advantage since the exact smoothness of the solution map

$$(\mu_0, b, \sigma) \mapsto (x \mapsto \mu_T(x))$$

may be difficult to obtain, even when μ_0, b, σ are given analytically. At an abstract level, if we assume that we can observe $(X_t^1, \dots, X_t^N)_{t \in [0, T]}$ exactly, a statistical theory has been recently proposed to recover $\mu_T(x)$ in [18], a key reference for the paper. However, the fact that we need to simulate first a proxy of the idealised data via a system of interacting Euler schemes needs a significant adjustment in order to obtain precise error guarantees. For other recent and alternate approaches based on projection-based particle method and the particle regression Monte Carlo method, we refer to [4, 5] and the references therein.

1.2. Results and organisation of the paper

In Section 2.1 we give the precise conditions on the data (μ_0, σ, b) . We need a smooth and sub-Gaussian integrable initial condition μ_0 in order to guarantee the existence of a classical solution $\mu_t(x)$ to the Fokker–Planck equation (1.1) and sub-Gaussianity of the process $(X_t)_{t \in [0, T]}$ that solves the McKean–Vlasov equation (1.2). The coefficients $b(t, x, \mu_t)$, $\sigma(t, x)$ are smooth functions of (t, x) and σ is uniformly elliptic, following the conditions of Gobet and Labart [22], where the expression “small time” in the title of [22] refers to a small time neighborhood rather than to a small value of T .

In Section 2.2 we construct the estimator $\widehat{\mu}_T^{N, h, \eta}(x)$ of (1.5) via a regular kernel K of order ℓ , see Definition 2.1. It depends on the size N of the particle system, the step $h > 0$ of the Euler scheme and the statistical smoothing parameter η and the order of the kernel ℓ . Abbreviating $K_\eta = \eta^{-d}K(\eta^{-1}\cdot)$ for $\eta > 0$, we have the following decomposition for the simulation error:

$$\widehat{\mu}_T^{N, h, \eta}(x) - \mu_T(x) = \int_{\mathbb{R}^d} K_\eta(x - \cdot) d(\bar{\mu}_T^{N, h} - \bar{\mu}_T^h) + K_\eta \star (\bar{\mu}_T^h - \mu_T)(x) + K_\eta \star \mu_T(x) - \mu_T(x),$$

where \star denotes convolution, $\bar{\mu}_T^{N, h} = N^{-1} \sum_{n=1}^N \delta_{\bar{X}_T^{n, h}}$ and $\bar{\mu}_T^h$ is the probability distribution of the continuous Euler scheme at time $t = T$, constructed in (2.4) below. In turn, the study of the error splits into three terms, with only the first one being stochastic. We prove in Proposition 4.1 a Bernstein concentration inequality for the fluctuation of $\bar{\mu}_T^{N, h}(dy) - \bar{\mu}_T^h(dy)$. In contrast to [40], it has the advantage to encompass test functions that may behave badly in Lipschitz norm, while they are stable in L^1 like a kernel K_η for small values of η . It reads, for an arbitrary bounded test function φ :

$$\mathbb{P}\left(\int_{\mathbb{R}^d} \varphi d(\bar{\mu}_T^{N, h} - \bar{\mu}_T^h) \geq \varepsilon\right) \leq \kappa_1 \exp\left(-\frac{\kappa_2 N \varepsilon^2}{|\varphi|_{L^2(\mu_T)}^2 + |\varphi|_\infty \varepsilon}\right), \text{ for every } \varepsilon \geq 0,$$

with explicitly computable constants κ_1 and κ_2 that only depend on T and the data (μ_0, b, σ) . The proof is based on a change of probability argument via Girsanov’s theorem. It builds on ideas developed in [18] and [32], but it requires nontrivial improvements in the case of approximating schemes, in particular fine estimates in Wasserstein distance between the distributions $\bar{\mu}_T^h$ and μ_T , thanks to Liu [37]. When the measure argument in the drift is nonlinear, we impose some smoothness in terms of linear differentiability, see Assumption A4. One limitation of this approach is that we can only encompass a measure term in the drift but not in the diffusion coefficient. The second approximation term $K_\eta \star (\bar{\mu}_T^h(x) - \mu_T(x))$ is managed via the uniform estimates of Gobet and Labart [22] of $\bar{\mu}_T^h(x) - \mu_T(x)$ so that we can ignore the effect of K_η as $\eta \rightarrow 0$ by the L^1 -stability $|K_\eta|_{L^1} = |K|_{L^1}$. The final bias term $K_\eta \star \mu_T(x) - \mu_T(x)$ is controlled by standard kernel approximation.

In Theorem 2.6 we give our main result, namely a deviation probability for the error $\widehat{\mu}_T^{N, h, \eta}(x) - \mu_T(x)$, provided h and η are small enough. In Theorem 2.7, we give a quantitative bound for the

expected error of order $p \geq 1$ that is further optimised in Corollary 2.8 when $x \mapsto \mu_T(x)$ is k -times continuously differentiable. We obtain a (normalised) error of order $N^{-k/(2k+d)} + h$, which combines the optimal estimation error in nonparametric statistics with the optimal strong error of the Euler scheme. Finally, we construct in Theorem 2.9 a variant of the Goldenshluger–Lepski’s method that automatically selects an optimal data-driven smoothing parameter η without requiring any prior knowledge on the smoothness of the solution. As explained in Section 1.1, our approach suggests a natural way to take advantage of adaptive statistical methods in numerical simulation, when even qualitative information about the object to simulate are difficult to obtain analytically.

We numerically implement our method on several examples in Section 3. We show in particular that it seems beneficial to use high-order kernels (i.e. with many vanishing moments) rather than simply Gaussian ones or more generally even kernels that only have one vanishing moment. This is not a surprise from a statistical point of view, since $x \mapsto \mu_T(x)$ is usually very smooth. Also, the data-driven choice of the bandwidth seems useful even in situations where our assumptions seem to fail (like for the Burgers equation, see in particular Section 3.3). The proofs are delayed until Section 4.

2. Main results

2.1. Notation and assumptions

For $x \in \mathbb{R}^d$, $|x|^2 = x \cdot x^\top$ denotes the Euclidean norm. We endow \mathcal{P}_1 with the 1-Wasserstein metric

$$\mathcal{W}_1(\mu, \nu) = \inf_{\rho \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| \rho(dx, dy) = \sup_{|\varphi|_{\text{Lip}} \leq 1} \int_{\mathbb{R}^d} \varphi d(\mu - \nu),$$

where $\Gamma(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν , and $|\varphi|_{\text{Lip}} = \sup_{x \neq y} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}$ is the Lipschitz semi-norm of φ .

All \mathbb{R}^d -valued functions f defined on $[0, T]$, \mathbb{R}^d , \mathcal{P}_1 (or product of those) are implicitly measurable for the Borel σ -field induced by the (product) topology and written componentwise as $f = (f^1, \dots, f^d)$, where the f^i are real-valued. Following [22], we say that a function $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is $\mathcal{C}_b^{k,l}$ for integers $k, l \geq 0$ if the real-valued functions f^i in the representation $f = (f^1, \dots, f^d)$ are all bounded and continuously differentiable with uniformly bounded derivatives w.r.t. t and x up to order k and l respectively. We write \mathcal{C}_b^k (or \mathcal{C}_b^l) if f is $\mathcal{C}_b^{k,l}$ and only depends on the time (or space) variable.

For $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, a σ -finite measure ν on \mathbb{R}^d and $1 \leq p < \infty$, we set

$$|\varphi|_{L^p(\nu)} = \left(\int_{\mathbb{R}^d} |\varphi(x)|^p \nu(dx) \right)^{1/p}, \quad |\varphi|_{L^p} = \left(\int_{\mathbb{R}^d} |\varphi(x)|^p dx \right)^{1/p}, \quad |\varphi|_\infty = \sup_{x \in \mathbb{R}^d} |\varphi(x)|.$$

Assumption A1. *The law $\mathcal{L}(X_0) = \mu_0(dx)$ in (1.2) or equivalently in (1.1) is sub-Gaussian, that is, $\int_{\mathbb{R}^d} \exp(\gamma|x|^2) \mu_0(dx) < \infty$ for some $\gamma > 0$.*

The strong integrability assumption of μ_0 (together with the subsequent smoothness properties of b and σ) will guarantee that μ_t is sub-Gaussian for every $t > 0$. This is the gateway to our change of probability argument in the Bernstein inequality of the Euler scheme in Proposition 4.1 below.

Assumption A2. *The diffusion matrix $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is uniformly elliptic, $\mathcal{C}_b^{1,3}$ and $\partial_t \sigma$ is $\mathcal{C}_b^{0,1}$.*

By uniform ellipticity, we mean $\inf_{t,x} \xi^\top c(t, x) \xi \geq C_\sigma |\xi|^2$ for every $\xi \in \mathbb{R}^d$ and some $C_\sigma > 0$, where $c = \sigma \sigma^\top$.

As for the drift $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}^d$, we separate the case whether the dependence in the measure argument is linear or not. A nonlinear dependence is technically more involved and this case is postponed to Section 2.4.

Assumption A3 (Linear representation of the drift.). *We have*

$$b(t, x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(t, x, y) \mu(dy),$$

with $(x, y) \mapsto \tilde{b}(t, x, y)$ Lipschitz continuous (uniformly in t), $x \mapsto \tilde{b}(t, x, y) \in C_b^3$ (uniformly in t and in y), $y \mapsto \tilde{b}(t, x, y) \in C_b^2$ (uniformly in t and in x), $(t, x, y) \mapsto \partial_t \tilde{b}(t, x, y)$ continuous and bounded and $\sup_{t \in [0, T]} |\tilde{b}(t, 0, 0)| < \infty$.

Assumption A3 implies that the function $(t, x, \mu) \mapsto b(t, x, \mu)$ is Lipschitz continuous, that is, there exists a constant $|b|_{\text{Lip}} > 0$ such that for every $(t, x, \mu), (s, y, \nu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_1$,

$$|b(t, x, \mu) - b(s, y, \nu)| \leq |b|_{\text{Lip}}[|t - s| + |x - y| + \mathcal{W}_1(\mu, \nu)], \quad (2.1)$$

where the conclusion follows from the Kantorovich duality for the Wasserstein distance (see, e.g., [15, Corollary 5.4]). As detailed in the proof of Proposition 4.3 below, Assumption A3 also implies that $(t, x) \mapsto b(t, x, \mu_t)$ is $C_b^{1,3}$. This together with Assumption A2 on the diffusion coefficient enables us to obtain a sharp approximation of the density of the Euler scheme of (1.2), a key intermediate result. This is possible thanks to the result of Gobet and Labart [22], which, to our knowledge, is the best result in that direction. See also [3, 26, 30] for similar results under stronger assumptions.

Throughout the paper, we use the same symbol μ_t to denote both the probability distribution and its density (the same convention applies to $\bar{\mu}_t^h$, see Definition 2.3). Specifically, $\mu_t(dx)$ denotes the distribution, while $\mu_t(x)$ denotes its density with respect to the Lebesgue measure. This slight abuse of notation will be maintained throughout. Regarding constants, we denote by \mathfrak{M} generic constants that may vary from line to line and depend only on the model parameters μ_0, b, σ, T, d . When the precise dependence is needed, we make it explicit in order to emphasize the links between results.

2.2. Construction of an estimator of $\mu_T(x)$

We pick an integer $M \in \mathbb{N} \setminus \{0\}$ for time discretisation and $h := \frac{T}{M}$ as time step. We set $t_m := m \cdot h$, $m \in \{0, \dots, M\}$.

For $1 \leq n \leq N$, let $B^n = (B_t^n)_{t \in [0, T]}$ be N independent d -dimensional Brownian motions. We construct N interacting processes $(\bar{X}_t^{1,h}, \dots, \bar{X}_t^{N,h})_{t \in [0, T]}$ via the following Euler schemes

$$\begin{cases} \bar{X}_{t_{m+1}}^{n,h} = \bar{X}_{t_m}^{n,h} + h \cdot b(t_m, \bar{X}_{t_m}^{n,h}, \bar{\mu}_{t_m}^{N,h}) + \sqrt{h} \cdot \sigma(t_m, \bar{X}_{t_m}^{n,h}) Z_{m+1}^n, & 1 \leq n \leq N, \\ Z_{m+1}^n := \frac{1}{\sqrt{h}} (B_{t_{m+1}}^n - B_{t_m}^n), \quad \bar{\mu}_{t_m}^{N,h} := \frac{1}{N} \sum_{n=1}^N \delta_{\bar{X}_{t_m}^{n,h}}, \\ \mathcal{L}(\bar{X}_0^{1,h}, \dots, \bar{X}_0^{N,h}) = \mu_0^{\otimes N}, \end{cases} \quad (2.2)$$

with $(\bar{X}_0^{1,h}, \dots, \bar{X}_0^{N,h})$ independent of $(B^n)_{1 \leq n \leq N}$. They are appended with their continuous version: for every $0 \leq m \leq M - 1$, $t \in [t_m, t_{m+1})$ and $1 \leq n \leq N$:

$$\bar{X}_t^{n,h} = \bar{X}_{t_m}^{n,h} + (t - t_m) b(t_m, \bar{X}_{t_m}^{n,h}, \bar{\mu}_{t_m}^{N,h}) + \sigma(t_m, \bar{X}_{t_m}^{n,h}) (B_t^n - B_{t_m}^n).$$

Definition 2.1. Let $\ell \geq 0$ be an integer. A ℓ -regular kernel is a bounded function $K : \mathbb{R}^d \rightarrow \mathbb{R}$ such that for $k = 0, \dots, \ell$, $\int_{\mathbb{R}^d} |x|^k |K(x)| dx < +\infty$ and

$$\int_{\mathbb{R}^d} x_{[1]}^k K(x) dx = \dots = \int_{\mathbb{R}^d} x_{[d]}^k K(x) dx = \mathbf{1}_{\{k=0\}} \quad \text{with } x = (x_{[1]}, \dots, x_{[d]}) \in \mathbb{R}^d. \quad (2.3)$$

Remark 2.2. A standard construction of ℓ -order kernels is discussed in [49], see also the classical textbook by Tsybakov [53]. In the numerical examples of the paper, we implement in dimension $d = 1$ the Gaussian high-order kernels described in Wand and Schucany [54]. Given a ℓ -regular kernel \mathfrak{h} in dimension 1, a multivariate extension is readily obtained by tensorisation: for $x = (x_{[1]}, \dots, x_{[d]}) \in \mathbb{R}^d$, set $K(x) = \mathfrak{h}^{\otimes d}(x) = \prod_{l=1}^d \mathfrak{h}(x_{[l]})$, so that (2.3) is satisfied.

Finally, we pick a ℓ -regular kernel K , a bandwidth $\eta > 0$ and set

$$\widehat{\mu}_T^{N,h,\eta}(x) := N^{-1} \sum_{n=1}^N \eta^{-d} K(\eta^{-1}(x - \bar{X}_T^{n,h}))$$

for our estimator of $\mu_T(x)$. It is thus specified by the kernel K , the Euler scheme time step h , the number of particles N and the statistical smoothing parameter η .

2.3. Convergence results

Under Assumptions A1, A2 and (A3 (or later A4)), (1.1) admits a unique classical solution $(t, x) \mapsto \mu_t(x)$, see e.g. the classical textbook [6], Corollary 8.2.2. Our first result is a deviation inequality for the error $\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)$. We need some further notation.

Definition 2.3. The abstract Euler scheme relative to the McKean–Vlasov equation (1.2) is defined as

$$\begin{cases} \bar{X}_{t_{m+1}}^h = \bar{X}_{t_m}^h + h \cdot b(t_m, \bar{X}_{t_m}^h, \mu_{t_m}) + \sqrt{h} \cdot \sigma(t_m, \bar{X}_{t_m}^h) Z_{m+1}, \\ Z_{m+1} = \frac{1}{\sqrt{h}}(B_{t_{m+1}} - B_{t_m}) \sim \mathcal{N}(0, 1), \\ \mathcal{L}(\bar{X}_{t_0}^h) = \mu_0, \end{cases} \quad (2.4)$$

appended with its continuous version: for every $0 \leq m \leq M-1$, $t \in [t_m, t_{m+1})$ and $1 \leq n \leq N$:

$$\bar{X}_t^h = \bar{X}_{t_m}^h + (t - t_m)b(t_m, \bar{X}_{t_m}^h, \mu_{t_m}) + \sigma(t_m, \bar{X}_{t_m}^h)(B_t - B_{t_m}).$$

We set $\bar{\mu}_t^h = \mathcal{L}(\bar{X}_t^h)$. The term *abstract* for this version of the Euler scheme comes from its explicit dependence upon μ_{t_m} , that prohibits its use in practical simulation. It however appears as a natural approximating quantity.

Definition 2.4. For $\varepsilon > 0$, the ε -accuracy of the abstract Euler scheme is

$$h^*(\varepsilon) = \sup \left\{ h > 0, \sup_{x \in \mathbb{R}^d} |\bar{\mu}_T^h(x) - \mu_T(x)| \leq \varepsilon \right\}, \quad (2.5)$$

with the convention $\sup \emptyset = \infty$.

Proposition 4.3 below, relying on the estimates of [22], implies that $\sup_{x \in \mathbb{R}^d} |\bar{\mu}_T^h(x) - \mu_T(x)|$ is of order h . Therefore $h^*(\varepsilon)$ is well defined and positive for every $\varepsilon > 0$.

We further abbreviate $K_\eta(x) = \eta^{-d} K(\eta^{-1}x)$ for $\eta > 0$.

Definition 2.5. The bias (relative to a ℓ -regular kernel K) of a function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ at x and scale $\eta > 0$ is

$$\mathcal{B}_\eta(\varphi, x) = \sup_{0 < \eta' \leq \eta} \left| \int_{\mathbb{R}^d} K_{\eta'}(x - y) \varphi(y) dy - \varphi(x) \right|$$

and for $\varepsilon > 0$, the ε -accuracy of the bias of φ is

$$\eta^*(\varepsilon, \varphi) = \sup \{ \eta > 0, \mathcal{B}_\eta(\varphi, x) \leq \varepsilon \}. \quad (2.6)$$

Note that whenever φ is continuous in a vicinity of x , we have $\mathcal{B}_\eta(\varphi, x) \rightarrow 0$ as $\eta \rightarrow 0$, and $\eta^*(\varepsilon, \varphi)$ is well-defined and positive for every $\varepsilon > 0$.

Theorem 2.6. *Work under Assumptions A1, A2 and A3. Let $x \in \mathbb{R}^d$, $N \geq 2$, $\varepsilon > 0$ and $\mathfrak{C} > 0$. For any $0 < h < \min\left(h^*\left(\frac{1}{3}|K|_{L^1}^{-1}\varepsilon\right), \mathfrak{C}N^{-1}\right)$ and $0 < \eta < \eta^*\left(\frac{1}{3}\varepsilon, \mu_T\right)$, we have:*

$$\mathbb{P}\left(|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)| \geq \varepsilon\right) \leq 2\kappa_1 \exp\left(-\frac{\kappa_2 N \varepsilon^2}{|K_\eta(x - \cdot)|_{L^2(\mu_T)}^2 + |K_\eta|_\infty \varepsilon}\right), \quad (2.7)$$

where κ_1 is a constant depending on \mathfrak{C} and \mathfrak{M} , and κ_2 depends only on \mathfrak{M} .

Several remarks are in order:

- (1) Let $x \in \mathcal{K} \subset \mathbb{R}^d$, where \mathcal{K} is an arbitrary compact. Then,

$$|K_\eta(x - \cdot)|_{L^2(\mu_T)}^2 \leq \sup_{(x,y) \text{ s.t. } x \in \mathcal{K}, x-y \in \text{Supp}(K)} \mu_T(y) |K_\eta(x - \cdot)|_{L^2}^2 = C(\mu_T, \mathcal{K}, K) \eta^{-d}$$

and $|K_\eta|_\infty = \eta^{-d} |K|_\infty$, therefore (2.7) rather reads

$$\mathbb{P}\left(|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)| \geq \varepsilon\right) \leq c_1 \exp\left(-c_2 N \eta^d \frac{\varepsilon^2}{1 + \varepsilon}\right),$$

up to constants c_1 and c_2 that only depend on \mathfrak{C} , \mathcal{K} , T , the kernel K and the model constant \mathfrak{M} .

- (2) The factor $\frac{1}{3}$ in the upper bound of h and η can be replaced by an arbitrary constant in $(0, \frac{1}{2})$ by modifying the union bound argument (4.13) in the proof. This modification only affects the multiplicative constant in the prefactor of the convergence rate in N of the forthcoming Theorem 2.7.
- (3) The estimate (2.7) gives an exponential bound of the form $\exp(-cN\eta^d\varepsilon^2)$ for the behaviour of the error $|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)|$ for small ε , provided h and η are sufficiently small. This is quite satisfactory in terms of statistical accuracy, for instance if one wants to implement confidence bands: for any risk level α , with probability bigger than $1 - \alpha$, the error $|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)|$ is controlled by a constant times $\sqrt{N^{-1}\eta^d \log \frac{1}{\alpha}}$.
- (4) We need Nh bounded. This stems from the key estimate (4.14) in the proof of Proposition 4.2 where we need to control N times the squared Wasserstein distance between the solution and the distribution of its Euler scheme which is of order h , in order to apply Novikov's criterion for an appropriate change of measure, see the proof for the details.
- (5) For the precise definitions of the constants κ_1 and κ_2 in (2.7), we refer the reader to Proposition 4.1. Note that although κ_2 depends only on \mathfrak{M} , we retain the notation κ_2 for the subsequent discussion (see, e.g., the remarks following Theorem 2.9), which concerns the relation between the constants.

Theorem 2.6 does not give any quantitative information about the accuracy in terms of h, N and η . Our next result gives an explicit upper bound for the expected error of order $p \geq 1$.

Theorem 2.7. *Work under Assumptions A1, A2 and A3. Let $\mathfrak{C} > 0$ and assume $Nh \leq \mathfrak{C}$. For every $p \geq 1$ and $x \in \mathcal{K} \subset \mathbb{R}^d$, where \mathcal{K} is an arbitrary compact, we have*

$$\mathbb{E}\left[|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)|^p\right] \leq \kappa_3 \left(\mathcal{B}_\eta(\mu_T, x)^p + (N\eta^d)^{-p/2} + h^p\right). \quad (2.8)$$

where \mathcal{B}_η is defined in Definition 2.5, and $\kappa_3 > 0$ depends on \mathfrak{C} , p , \mathcal{K} , $|K|_{L^2}$, $|K|_\infty$ and the model constant \mathfrak{M} .

If $z \mapsto \mu_T(z)$ is k -times continuously differentiable in a vicinity of x , then, for a regular kernel of order $(k-1)_+$, we have $\mathcal{B}_\eta(\mu_T, x) \lesssim \eta^k$, see for instance the proof of Corollary 2.8. This enables one to optimise the choice of the bandwidth η :

Corollary 2.8. *Assume that $x \mapsto \mu_T(x)$ is \mathcal{C}_b^k for some $k \geq 1$ and that K is ℓ -regular, with $\ell \geq 0$. In the setting of Theorem 2.7, the optimisation of the bandwidth choice $\eta = N^{-1/(2\min(k,\ell+1)+d)}$ yields the explicit error rate*

$$\mathbb{E}\left[|\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)|^p\right] \leq \kappa_4 \left(N^{-\min(k,\ell+1)p/(2\min(k,\ell+1)+d)} + h^p\right), \quad (2.9)$$

for some $\kappa_4 > 0$ depending on \mathfrak{C} , p , ℓ , \mathcal{K} , $|K|_{L^2}$, $|K|_\infty$ and the model constant \mathfrak{M} .

Some remarks:

- (1) The dependence of κ_3 and κ_4 on (μ_0, b, σ) can be computed explicitly via the bounds that appear in Assumptions A1, A2 and A3.
- (2) Corollary 2.8 improves the result of [1, Theorem 3.1] for the strong error of the classical solution $\mu_T(x)$ in the following way: in [1], the authors restrict themselves to the one-dimensional case $d = 1$ with $p = 1$ in the loss and take a Gaussian kernel for the density estimation, with bandwidth $h = \frac{T}{M}$ as in the time discretisation. They obtain the rate (in dimension $d = 1$)

$$N^{-1/2}h^{-1/2-\epsilon} + h,$$

for arbitrary $\epsilon > 0$ (up to an inflation in the constant when ϵ vanishes), and this has to be compared in our case to the rate (2.9) which gives

$$N^{-\frac{k}{2k+1}} + h = N^{-\frac{1}{2}}N^{\frac{1}{2(2k+1)}} + h,$$

which yields possible improvement depending on the regime for h , namely whenever $N \ll h^{-(2k+1)}$ which is less demanding as k increases. The essence of the improvement compared to [1] is that we make an optimal bias-variance analysis based on nonparametric statistics techniques. In particular, the introduction of kernels that oscillate more than the Gaussian kernel enables us to obtain an error bound linked to the smoothness parameter k that can be very large whenever the classical solution $x \mapsto \mu_t(x)$ is smooth.

One defect of the upper bound (2.9) is that the optimal choice of η depends on the analysis of the bias $\mathcal{B}_\eta(\mu_T, x)^p$, and more specifically on the smoothness k , and that quantity is usually unknown or difficult to compute. Indeed, the information that $x \mapsto \mu_T(x)$ is k -times continuously differentiable with the best possible k is hardly tractable from the data (μ_0, b, σ) , essentially due to the nonlinearity of the Fokker–Planck equation (1.1). Finding an optimal η without prior knowledge on the bias is a long standing issue in nonparametric statistics. One way to circumvent this issue is to select η depending on the stochastic particle system $(\bar{X}_T^{1,h}, \dots, \bar{X}_T^{N,h})$ itself, in order to optimise the trade-off between the bias and the variance term. While this is a customary approach in statistics, to the best of our knowledge, this is not the case in numerical simulations. It becomes particularly suitable in the case of nonlinear McKean–Vlasov type models.

We adapt a variant of the classical Goldenshluger–Lepski’s method [23, 24, 25], and refer the reader to classical textbooks such as [21]. See also the illuminating paper by [33] that gives a good insight about ideas around bandwidth comparison and data driven selection. For simplicity, we focus on controlling the error for $p = 2$. Fix $x \in \mathbb{R}^d$. Pick a finite set $\mathcal{H} \subset [N^{-1/d}(\log N)^{2/d}, 1]$, and define

$$A_\eta^N(T, x) = \max_{\eta' \leq \eta, \eta' \in \mathcal{H}} \left\{ \left(\hat{\mu}_T^{N,h,\eta}(x) - \hat{\mu}_T^{N,h,\eta'}(x) \right)^2 - (\mathbf{V}_\eta^N + \mathbf{V}_{\eta'}^N) \right\}_+,$$

where $\{x\}_+ = \max(x, 0)$ and

$$\mathbf{V}_\eta^N = \varpi |K|_{L^2}^2 (\log N) N^{-1} \eta^{-d}, \quad \varpi > 0. \tag{2.10}$$

Let

$$\hat{\eta}^N(T, x) \in \operatorname{argmin}_{\eta \in \mathcal{H}} (A_\eta^N(T, x) + \mathbf{V}_\eta^N). \tag{2.11}$$

We then have a so-called *oracle inequality*, following the standard terminology in nonparametric statistics. It means that our error bound is comparable to the optimal trade off between bias and variance in our method, namely $\min_{\eta \in \mathcal{H}} (\mathcal{B}_\eta(\mu_T, x)^2 + N^{-1} \eta^{-d})$ which is unknown to us, since it explicitly depends on μ_T in a vicinity of x . We also have an additional term of order h^2 linked to the Euler scheme. More precisely, we have

Theorem 2.9. *Work under Assumptions A1, A2 and A3. For every $x \in \mathcal{K} \subset \mathbb{R}^d$, where \mathcal{K} is an arbitrary compact, we have*

$$\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \mu_T(x) \right)^2 \right] \leq \kappa_5 \left(\min_{\eta \in \mathcal{H}} \left(\mathcal{B}_\eta(\mu_T, x)^2 + N^{-1} \eta^{-d} \log N \right) + h^2 \right), \quad (2.12)$$

for some constant $\kappa_5 > 0$ depending on \mathcal{K} , $|K|_\infty$, $\text{Supp}(K)$ and the model constant \mathfrak{M} , as soon as $\text{Card}(\mathcal{H}) \leq N$, ϖ is large enough, and $\eta^d \geq N^{-1} \log N \frac{3 \log 2 |K|_\infty^2}{2|K|_L^2}$.

Some remarks:

- (1) Up to an unavoidable $\log N$ -factor, known as the Lepski–Low phenomenon (see [36, 39]) in statistics, we thus have a data-driven smoothing parameter $\widehat{\eta}^N(T, x)$ that automatically achieves the optimal bias-variance balance, without affecting the effect of the Euler discretisation step h .
- (2) The detailed constraint on ϖ is provided in the proof of Theorem 2.9. A limitation of the method is the choice of the pre-factor ϖ in the bandwidth selection procedure, which depends on upper bound of μ_T locally around x , and more worryingly, on the constant κ_2^{-1} of Proposition 4.1 below which is quite large. In practice, and this is universal to all smoothing statistical methods, we have to adjust a specific numerical protocol, see Section 3 below.

When we translate this result in terms of number of derivatives for $x \mapsto \mu_T(x)$, we obtain the following adaptive estimation result.

Corollary 2.10. *Assume that $x \mapsto \mu_T(x)$ is \mathcal{C}_b^k for some $k \geq 1$ and that K is ℓ -regular. Specify the oracle estimator $\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x)$ with*

$$\mathcal{H} = \{(N/\log N)^{-1/(2m+d)}, m = 1, \dots, \ell + 1\}.$$

In the setting of Theorem 2.9, for every $x \in \mathcal{K} \subset \mathbb{R}^d$, where \mathcal{K} is an arbitrary compact, we have

$$\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \mu_T(x) \right)^2 \right] \leq \kappa_6 \left(\left(\frac{\log N}{N} \right)^{\frac{2 \min(k,\ell+1)}{2 \min(k,\ell+1)+d}} + h^2 \right),$$

for some $\kappa_6 > 0$ depending on \mathcal{K} , $|K|_\infty$, $\text{Supp}(K)$, k and the model constant \mathfrak{M} .

In practice, see in particular Section 3 below, if $x \mapsto \mu_T(x)$ is very smooth (and this is the case in particular if $x \mapsto b(t, x, \mu_t)$ is smooth and σ constant) we are limited in the rate of convergence by the order ℓ of the kernel. This shows in particular that it is probably not advisable in such cases to pick a Gaussian kernel for which we have the restriction $\ell = 1$.

2.4. Nonlinearity of the drift in the measure argument

In the case of a drift $(t, x, \mu) \mapsto b(t, x, \mu)$ with a nonlinear dependence in the measure argument μ , the assumptions are a bit more involved. For a smooth real-valued test function φ defined on \mathbb{R}^d , we set

$$\mathcal{A}_t \varphi(x) = b(t, x, \mu_t)^\top \nabla \varphi(x) + \frac{1}{2} \sum_{k,k'=1}^d c_{kk'}(t, x) \partial_{kk'}^2 \varphi(x), \quad (2.13)$$

where $c_{kk'}(t, x) = (\sigma \sigma^\top)_{kk'}(t, x)$, and that can also be interpreted as the generator of the associated nonlinear Markov process of $(X_t)_{t \in [0, T]}$ defined in (1.2). Following [15], we say that a mapping $f : \mathcal{P}_1 \rightarrow \mathbb{R}^d$ has a linear functional derivative if there exists $\delta_\mu f : \mathbb{R}^d \times \mathcal{P}_1 \rightarrow \mathbb{R}^d$ such that

$$f(\mu') - f(\mu) = \int_0^1 d\vartheta \int_{\mathbb{R}^d} \delta_\mu f(y, (1 - \vartheta)\mu + \vartheta\mu')(\mu' - \mu)(dy),$$

with $|\delta_\mu f(y', \mu') - \delta_\mu f(y, \mu)| \leq C(|y' - y| + \mathcal{W}_1(\mu', \mu))$ and $|\partial_y(\delta_\mu f(y, \mu') - \delta_\mu f(y, \mu))| \leq C\mathcal{W}_1(\mu', \mu)$, for some $C \geq 0$. We can iterate the process via mappings $\delta_\mu^\ell f : (\mathbb{R}^d)^\ell \times \mathcal{P}_1 \rightarrow \mathbb{R}^d$ for $\ell = 1, \dots, k$ defined recursively by $\delta_\mu^\ell f = \delta_\mu \circ \delta_\mu^{\ell-1} f$.

Assumption A4 (Nonlinear representation of the drift.). *For every $\mu \in \mathcal{P}_1$, $(t, x) \mapsto b(t, x, \mu)$ is $\mathcal{C}_b^{1,3}$. Moreover, for every x , $(t, y) \mapsto \mathcal{A}_t \delta_\mu b(t, x, y, \mu_t)$ exists and is continuous and bounded.*

Finally, $\mu \mapsto b(t, x, \mu)$ admits a k -linear functional derivative (with $k \geq d$ for $d = 1, 2$ and $k \geq d/2$ for $d \geq 3$) that admits the following representation

$$\delta_\mu^k b(t, x, (y_1, \dots, y_k), \mu) = \sum_{\mathcal{I}, m} \bigotimes_{j \in \mathcal{I}} (\delta_\mu^k b)_{\mathcal{I}, j, m}(t, x, y_j, \mu), \quad (2.14)$$

where the sum in \mathcal{I} ranges over subsets of $\{1, \dots, k\}$, the sum in m is finite and the mappings $(x, y) \mapsto (\delta_\mu^k b)_{\mathcal{I}, j, m}(t, x, y_j, \mu)$ are Lipschitz continuous, uniformly in t and μ .

Assumption A4 has two objectives: first comply with the property that $(t, x) \mapsto b(t, x, \mu_t)$ is $\mathcal{C}_b^{1,3}$ in order to apply the result of Gobet and Labart [22] in Proposition 4.3 below and second, provide a sufficiently smooth structure in (2.14) in order to implement the change of probability argument of Proposition 4.1. While (2.14) appears a bit *ad-hoc* and technical, it is sufficiently general to encompass drift of the form

$$b(t, x, \mu) = \sum_j F_j \left(t, x, \int_{\mathbb{R}^{q_2}} G_j \left(t, x, \int_{\mathbb{R}^d} H_j(t, x, z) \mu(dz), z' \right) \lambda_j(dz') \right)$$

for smooth mappings $F_j(t, x, \cdot) : \mathbb{R}^{q_3} \rightarrow \mathbb{R}^d$, $G_j(t, x, \cdot) : \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \rightarrow \mathbb{R}^{q_3}$, $H_j(t, x, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^{q_1}$ and positive measures λ_j on \mathbb{R}^{q_2} in some cases and combinations of these, see Jourdain and Tse [28]. Explicit examples where the structure of the drift is of the form (2.14) rather than the form in Assumption A3 are given for instance in [17, 27, 43, 45].

Theorem 2.11. *Work under Assumptions A1, A2 and A4. The results of Theorems 2.6, 2.7, 2.9 and Corollary 2.8, 2.10 remain valid, up to a modification of the constants κ_i , $i = 1, \dots, 6$.*

3. Numerical implementation

We investigate our simulation method on three different examples. The simulation code is available via [GitHub](https://github.com) (see: bit.ly/3Ur0miw).

- (1) *A linear interaction as in Assumption A3 of the form $\tilde{b}(t, x, y) = c(x - y)$.* This case is central to several applications (see the references in the introduction for instance) and has moreover the advantage to yield an explicit solution for $\mu_t(x)$, enabling us to accurately estimate the error of the simulation.
- (2) *A double layer potential with a possible singular shock in the common drift that “stresses” Assumption A3.* Although the solution is not explicit, the singularity enables us to investigate the automated bandwidth choice on repeated samples and sheds some light on the effect of our statistical adaptive method.
- (3) *The Burgers equation in dimension $d = 1$.* Although not formally within the reach Assumption A3, we may still implement our method. While we cannot provide theoretical guarantees, we again have an explicit solution for $\mu_t(x)$ and we can accurately measure the performance of our simulation method.

For simplicity, all the numerical experiments are conducted in dimension $d = 1$. The common findings of our numerical investigations can be summarised as follows: when the density solution $x \mapsto \mu_t(x)$ is smooth, high-order kernels (with vanishing moments beyond order 1) outperform classical kernels such as the Epanechnikov kernel or a standard Gaussian kernel as shown in Cases 1 and 3. Specifically, we implement high-order Gaussian based kernels as developed for instance in Wand and Schucany [54]. Table 1 below gives explicit formulae for K and $\ell = 1, 3, 5, 7, 9$ (Recall Remark 2.2 for the construction of ℓ -order kernels for $d > 1$). Otherwise, for instance when the transport term has a common Lipschitz continuous component (but not more regular than Lipschitz continuous), the automated bandwidth method tends to adapt the window to prevent over-smoothing, even with high-order kernels. Overall, we find in all three examples that implementing high-order kernels is beneficial for the quality of the simulation.

TABLE 3.1. Construction of kernels K of order $\ell = 1, 3, 5, 7, 9$, with $\phi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$, dimension $d = 1$, following [54].

order ℓ	kernel $K(x)$
1	$\phi(x)$
3	$\frac{1}{2}(3 - x^2)\phi(x)$
5	$\frac{1}{8}(15 - 10x^2 + x^4)\phi(x)$
7	$\frac{1}{48}(105 - 105x^2 + 21x^4 - x^6)\phi(x)$
9	$\frac{1}{384}(945 - 1260x^2 + 378x^4 - 36x^6 + x^8)\phi(x)$

3.1. The case of a linear interaction

We consider the simplest McKean–Vlasov SDE with linear interaction

$$dX_t = - \int_{\mathbb{R}} (X_t - x)\mu_t(dx)dt + dB_t, \quad \mathcal{L}(X_0) = \mathcal{N}(3, \frac{1}{2})$$

where $(B_t)_{t \in [0, T]}$ is a standard Brownian motion and $\mathcal{N}(\rho, \sigma^2)$ denotes the Gaussian distribution with mean ρ and variance σ^2 . We do not fulfill Assumption A3 here since $\tilde{b}(t, x, y) = x - y$ is not bounded. We nevertheless choose this model for simulation since it has an explicit solution μ_t as a stationary Ornstein–Uhlenbeck process, namely, abusing notation slightly, $\mu_t = \mathcal{L}(X_t) = \mathcal{N}(3, \frac{1}{2})$.

We implement the Euler scheme $(\bar{X}_t^{1, h}, \dots, \bar{X}_t^{N, h})_{t \in [0, T]}$ defined in (2.2) with $b(t, x, \mu) = - \int_{\mathbb{R}} (x - y)\mu(dy)$ and $\sigma(t, x) = 1$. We pick $T = 1$, $h = 10^{-2}T = 10^{-2}$, for several values of the system size $N = 2^7 = 128, 2^8 = 256, \dots, 2^{15} = 32768$. We then compute

$$\hat{\mu}_T^{N, h, \eta}(x) := N^{-1} \sum_{n=1}^N \eta^{-d} K(\eta^{-1}(x - \bar{X}_T^{n, h}))$$

as our proxy of $\mu_T(x)$ for $\ell = 1, 3, 5, 7, 9$ and the kernels given in Table 3.1. We pick $\eta = N^{-1/(2(\ell+1)+1)}$ according to Corollary 2.8 since $\mu_T \in \mathcal{C}^k$ for every $k \geq 1$. We repeat the experiment 30 times to obtain independent Monte-Carlo proxies $(\hat{\mu}_T^{N, h, \eta})_j(x)$ for $j = 1, \dots, 30$ and we finally compute the Monte-Carlo strong error $\mathcal{E}_N = \frac{1}{30} \sum_{j=1}^{30} \max_{x \in \mathcal{D}} |(\hat{\mu}_T^{N, h, \eta})_j(x) - \mu_T(x)|^2$, where \mathcal{D} is a uniform grid of 1000 points in $[0, 6]$. (The domain is dictated by our choice of initial condition $\mu_0 \sim \mathcal{N}(3, \frac{1}{2})$; the probability that a random variable with this distribution falls outside $[0, 6]$ is about 2.2×10^{-5} , which justifies this truncation.)

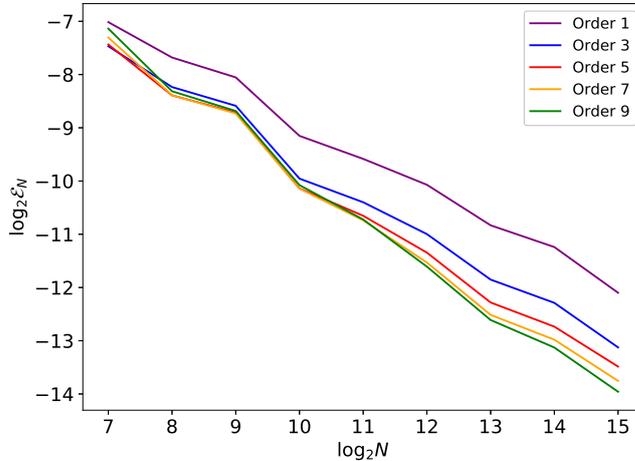


FIGURE 3.1. Monte-Carlo (for 30 repeated samples) strong error for different kernel orders: $\log_2 \mathcal{E}_N$ as a function of $\log_2 N$ for $\ell = 1$ (purple), $\ell = 3$ (blue), $\ell = 5$ (red), $\ell = 7$ (orange), $\ell = 9$ (green). We see that a polynomial error in N is compatible with the data.

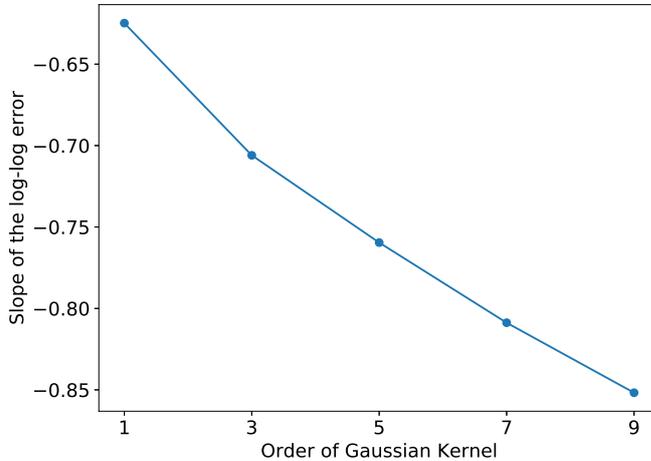


FIGURE 3.2. Least-square estimates of the slope α_ℓ of $\log_2 \mathcal{E}_N = \alpha_\ell \log_2 N + \text{noise}$ in a linear model representation. We plot α_ℓ as a function of the order ℓ of the kernel. We see that a higher order ℓ for the choice of the kernel systematically improves on the error rate, as predicted by the statistical bias-variance analysis.

In Figure 3.1, we display on a log-2 scale \mathcal{E}_N for $\ell = 1, 3, 5, 7, 9$. In Figure 3.2 we display the least-square estimates of the slope of each curve of Figure 3.1 according to a linear model. We thus have a proxy of the rate α_ℓ of the error $\mathcal{E}_N \approx N^{-\alpha_\ell}$ for different values of ℓ . We clearly see that higher-order kernels are better suited for estimating μ_t . This is of course no surprise from a statistical point of view, but this may have been overlooked in numerical probability simulations.

3.2. A double layer potential

We consider an interaction consisting of a smooth long range attractive and small range repulsive force, obtained as the derivative of double-layer Morse potential. Such models are commonly used (in their kinetic version) in swarming modelling, see for instance [7]. The corresponding McKean–Vlasov equation is

$$dX_t = \int_{\mathbb{R}} U'(X_t - x) \mu_t(dx) dt + dB_t, \quad \mathcal{L}(X_0) = \mathcal{N}(0, 1),$$

where we pick $U(x) = -\exp(-x^2) + 2\exp(-2x^2)$. The potential U and its derivative U' are displayed in Figure 3.3.

We implement the Euler scheme $(\bar{X}_t^{1,h}, \dots, \bar{X}_t^{N,h})_{t \in [0, T]}$ defined in (2.2) for coefficients $b(x, \mu) = \int_{\mathbb{R}} U'(x - y) \mu(dy)$ and $\sigma(t, x) = 1$. We pick $T = 1$, $h = 10^{-2}T = 10^{-2}$, for several values of the system size $N = 2^5 = 32, 2^6 = 64, \dots, 2^{16} = 65736$. We then compute

$$\hat{\mu}_T^{N,h, \hat{\eta}^N(T,x)}(x) := N^{-1} \sum_{n=1}^N \hat{\eta}^N(T, x)^{-d} K\left(\hat{\eta}^N(T, x)^{-1}(x - \bar{X}_T^{n,h})\right)$$

as our proxy of $\mu_T(x)$ for $\ell = 1, 3, 5, 7, 9$ according to K as in Table 3.1. The data-driven bandwidth $\hat{\eta}^N(T, x)$ is computed via the minimisation (2.11), for which one still needs to set the penalty parameter ϖ arising in the Goldenshluger–Lepski method, see (2.10) in particular. The grid is set as $\mathcal{H} = \{(N/\log N)^{-1/(2m+1)}, m = 1, \dots, \ell + 1\}$, see in particular Corollary 2.10 in order to mimic the oracle.

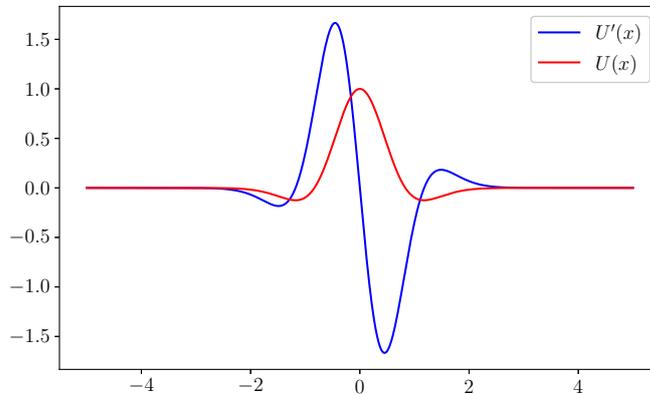


FIGURE 3.3. Plot of the potential U (red) and its derivative U' (blue).

In this setting, we do not have access to the exact solution $\mu_T(x)$; we nevertheless explore several numerical aspects of our method via the following experiments:

- Investigate the effect of the order ℓ of the kernel (for an *ad-hoc* choice of the penalty parameter ϖ). We know beforehand that the mapping $x \mapsto \mu_T(x)$ is smooth, and we obtain numerical evidence that a higher-order kernel gives better results by comparing the obtained $\hat{\mu}_T^{N,h, \hat{\eta}^N(T,x)}(x)$ for different values of ℓ as N increases in the following sense: for $N = 2^5$ particles, the estimates for high order kernels are closer to the estimates obtained for $N = 2^{10}$ than lower order kernels.

- Investigate the distribution of the data-driven bandwidth $\hat{\eta}^N(T, x)$, for repeated samples as x varies and for different values of the penalty ϖ over the grid $\mathcal{H} = \{(N/\log N)^{-1/(2m+1)}, m = 1, \dots, \ell + 1\}$. The estimator tends to pick the larger bandwidth with overwhelming probability, which is consistent with our prior knowledge that $x \mapsto \mu_T(x)$ is smooth.
- In order to exclude an artifact from the preceding experiment, we conduct a cross-experiment by perturbing the drift with an additional Lipschitz common force (but not smoother) that saturates our Assumption A3. This extra transport term lowers down the smoothness of $x \mapsto \mu_T(x)$ and by repeating the preceding experiment, we obtain a different distribution of the data-driven bandwidth $\hat{\eta}^N(T, x)$, advocating in favour of a coherent oracle procedure.

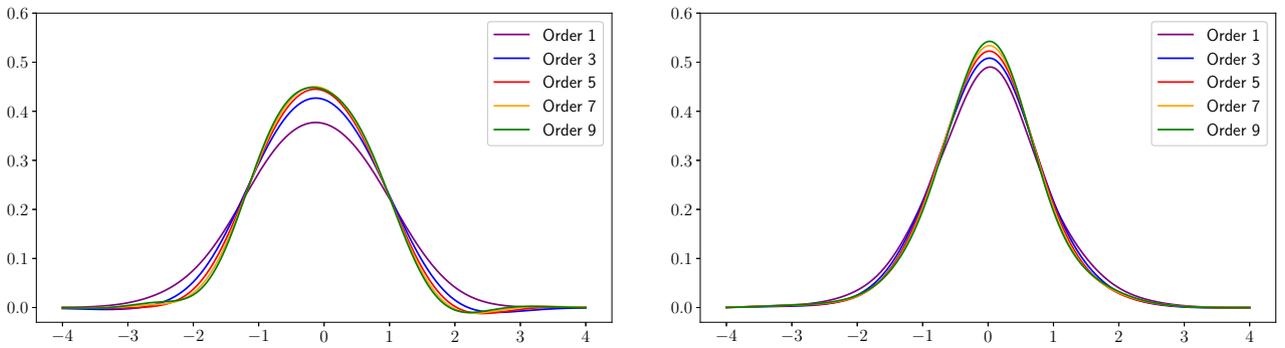


FIGURE 3.4. *The graph of $x \mapsto \hat{\mu}_t^{N,h,\hat{\eta}^N(t,x)}(x)$. The domain $x \in [-4, 4]$ is computed over a discrete grid of 2000 points, i.e. mesh $4 \cdot 10^{-3}$) for $N = 2^5$ (Left) and $N = 2^{10}$ (Right).*

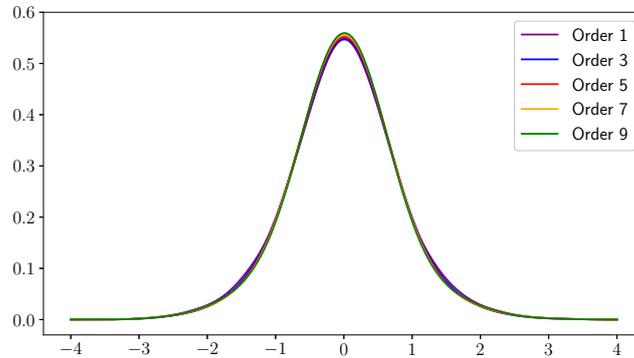


FIGURE 3.5. *Same experiment as in Figure 3.4 mimicking an asymptotic behaviour of the procedure for $N = 2^{15}$.*

The effect of the order ℓ of the kernel

We display in Figure 3.4 the graph of $x \mapsto \widehat{\mu}_T^{N,h,\widehat{\eta}^N}(T,x)$ for $T = 1$, $\ell = 1, 3, 5, 7, 9$ for $N = 2^5$ and $N = 2^{10}$. The tuning parameter in the choice of $\widehat{\eta}^N(T, x)$ is set to ¹ $\varpi = 23$. The same experiment is displayed in Figure 3.5 for $N = 2^{15}$.

The value $N = 2^{15}$ mimicks the asymptotic performance of the procedure as compared to $N = 2^5$ or $N = 2^{10}$. We observe that the effect of the order of the kernel ℓ is less pronounced. A visual comparison of Figure 3.4 and 3.5 suggests that a computation with higher order kernels always perform better, since the shape obtained for $\ell = 9$ for small values of N is closer to the asymptotic proxy $N = 2^{15}$ than the results obtained with smaller values of ℓ .

The distribution of the data-driven bandwidth

We pick several values for ϖ (namely $\varpi = 20^{-1}, 10^{-1}, 1, 10, 20$) and compute $\widehat{\eta}^N(T, x)$ accordingly for 1200 samples. Figure 3.6 displays the histogram of the $\widehat{\eta}^N(T, x)$ for $x \in [-4, 4]$ (discrete grid with mesh $8 \cdot 10^{-2}$) for $\ell = 3$ and $\ell = 5$. We observe that the distribution is peaked around large bandwidths at the far right of the spectrum of the histogram, with comparable results for $\ell = 7$ and $\ell = 9$. In Figure 3.7, we repeat the experiment for $\ell = 5$ over the restricted domain $[-\frac{1}{2}, \frac{1}{2}]$ (discrete grid, mesh 10^{-2}) where we expect the solution $\mu_T(x)$ to be more concentrated and see no significant difference. This result is in line with the statistical nonparametric estimation of a smooth signal (see e.g. the classical textbook [50]).

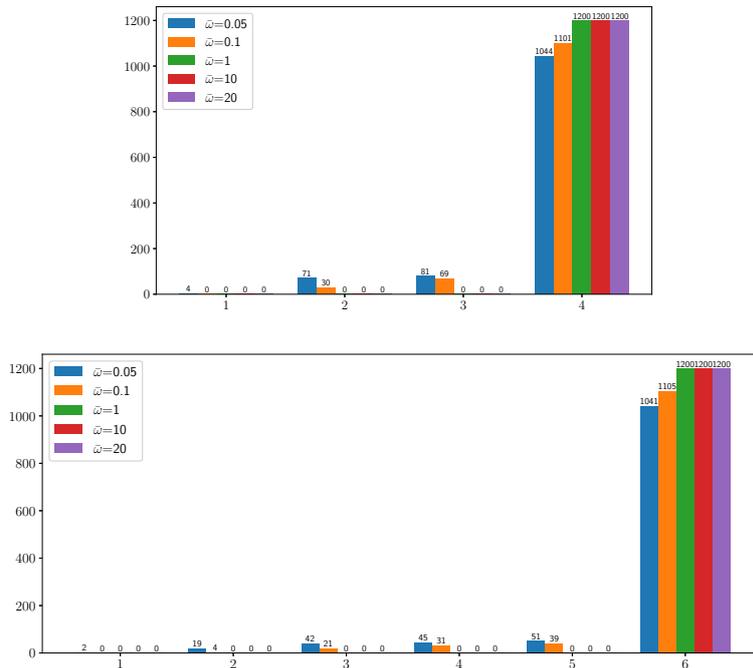


FIGURE 3.6. *Distribution of $\widehat{\eta}^N(T, x)$. The domain $x \in [-4, 4]$ is computed over a discrete grid of 100 points, i.e. mesh $8 \cdot 10^{-2}$), $N = 2^5, 2^6, \dots, 2^{16}$ for $\ell = 3$ (Top) and $\ell = 5$ (Bottom).*

¹This choice is quite arbitrary, the values of the data-driven bandwidths showing stability as soon as $\varpi \geq 20$.

Cross experiment by reducing the smoothness of $x \mapsto \mu_T(x)$

We repeat the previous experiment by adding a perturbation in the drift, via a common force $V'(x) = 2(1 - |x|)\mathbf{1}_{\{|x| \leq 1\}}$. The drift now becomes $b(x, \mu) = V'(x) + \int_{\mathbb{R}} U'(x - y)\mu(dy)$. The common transport term $V'(x)$ is no smoother than Lipschitz continuous thus reducing the smoothness of the solution $x \mapsto \mu_T(x)$. In the same experimental conditions as before, Figure 3.8 displays the distribution of $\hat{\eta}^N(T, x)$ and is to be compared with Figure 3.7. We observe that the distribution is modified, in accordance with the behaviour of the oracle bandwidth of a signal with lower order of smoothness. This advocates as further empirical evidence of the coherence of the method.

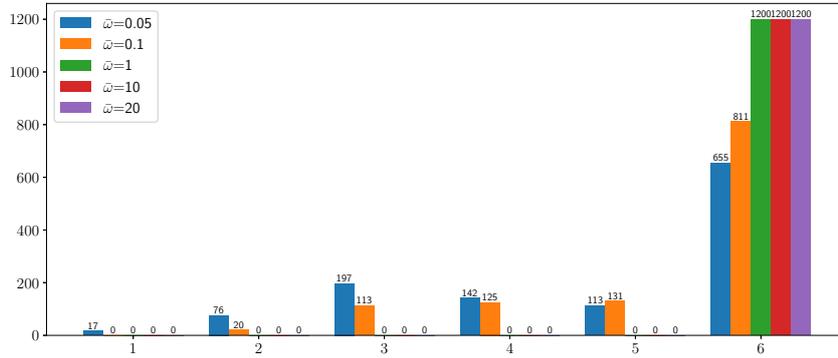


FIGURE 3.7. *Same experiment as in Figure 3.6 (Bottom) on the restricted domain $[-\frac{1}{2}, \frac{1}{2}]$ over a discrete grid of 100 points. The results are comparable with the experiment displayed in Figure 3.6 (Bottom).*

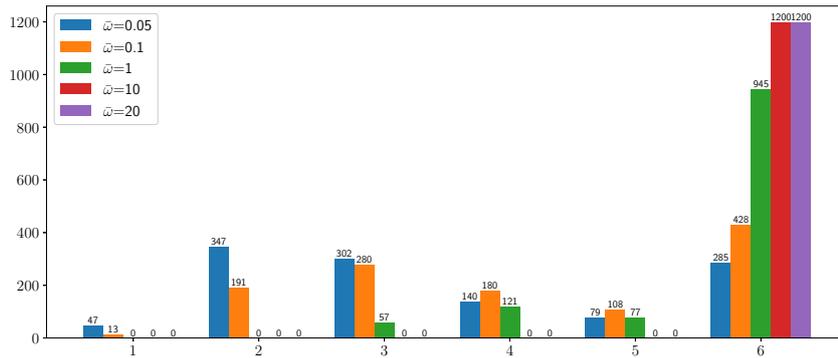


FIGURE 3.8. *Same experiment as in Figure 3.7 on the restricted domain $[-\frac{1}{2}, \frac{1}{2}]$ over a discrete grid of 100 points, when adding a common perturbation force $V'(x) = 2(1 - |x|)\mathbf{1}_{\{|x| \leq 1\}}$ in the drift. The effect on the empirical distribution of $\hat{\eta}^N(T, x)$ is in line with the behaviour of an oracle bandwidth that adjusts smaller bandwidths when the signal is less smooth.*

3.3. The Burgers equation in dimension $d = 1$

We consider here the following McKean–Vlasov equation

$$dX_t = \int_{\mathbb{R}} H(X_t - y) \mu_t(dy) dt + \sigma dB_t, \quad \mathcal{L}(X_0) = \delta_0,$$

with $\sigma = \sqrt{0.2}$, associated with the Burgers equation in dimension $d = 1$ for $H(x - y) = \mathbf{1}_{\{y \leq x\}}$. This model has already been introduced in [11], and the interacting particle system (2.2) matches Equation (2.3) of [11]. Although the discontinuity at $y = x$ rules out our Assumption A3, we may nevertheless implement our method, since a closed form formula is available for $\mu_t(x)$. More specifically, for $t > 0$, the cumulative distribution function of $\mu_t(x)$ is explicitly given by

$$M_t(x) = \frac{\int_0^\infty \exp\left(-\frac{1}{\sigma^2} \left(\frac{(x-y)^2}{2t} + y\right)\right) dy}{\int_{-\infty}^0 \exp\left(-\frac{1}{\sigma^2} \frac{(x-y)^2}{2t}\right) dy + \int_0^\infty \exp\left(-\frac{1}{\sigma^2} \left(\frac{(x-y)^2}{2t} + y\right)\right) dy}.$$

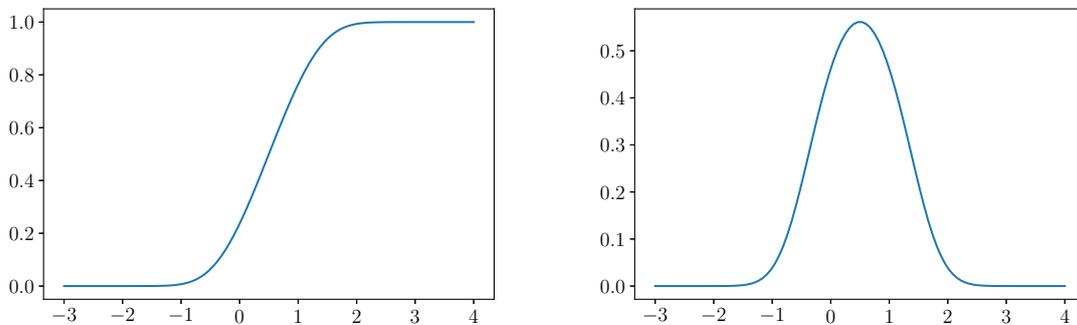


FIGURE 3.9. Plots of $x \mapsto M_T(x)$ (Left) and $x \mapsto \mu_T(x)$ (Right).

Figure 3.9 displays the graph of $x \mapsto M_T(x)$ and $x \mapsto \mu_T(x)$ for $T = 1$. We display in Figure 3.10 the reconstruction of $x \mapsto \mu_T(x)$ via $\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x)$ for $N = 2^{10}$ for several values of ℓ . The value $\ell = 9$ provides with the best fit, showing again the benefit of higher-order kernels compared to a standard Gaussian kernel (with $\ell = 1$). Similar results were obtained for other values of N . Yet, the distribution of $\widehat{\eta}^N(T, x)$ displayed in Figure 3.11 shows that our method tends to undersmooth the true density. The effect is less pronounced for $N = 2^{15}$. In any case, our numerical results are comparable with [11].

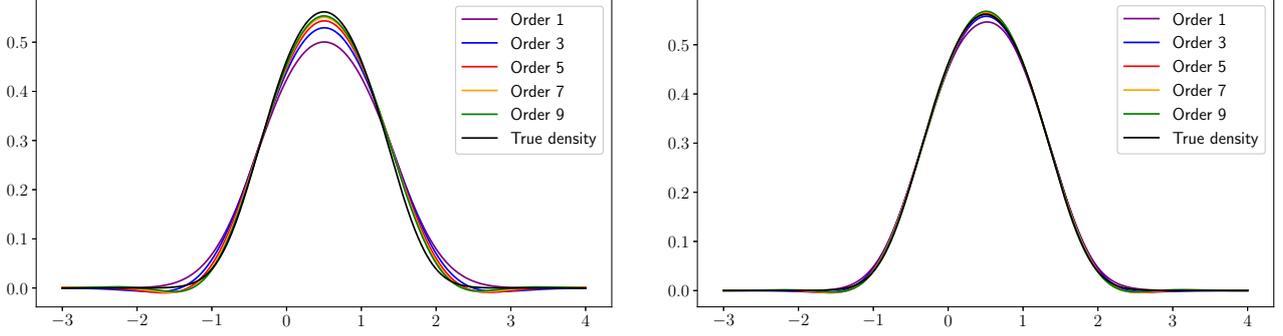


FIGURE 3.10. Reconstruction of μ_T by $\hat{\mu}_T^{N,h,\hat{\eta}^N(T,\cdot)}$ with $\varpi = 23$, for different kernel orders for $N = 2^{10}$ (Left) and $N = 2^{15}$ (Right). Higher order kernels outperform the reconstruction provided with standard Gaussian kernels ($\ell = 1$) for $N = 2^{10}$.

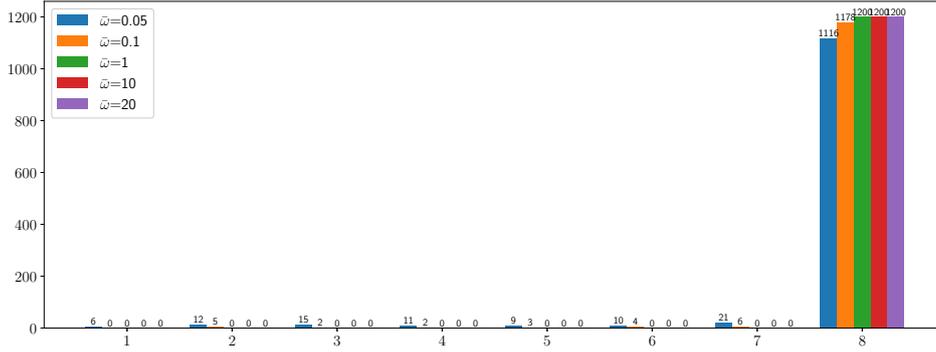


FIGURE 3.11. Distribution of $\hat{\eta}^N(T, x)$. The domain $x \in [-3, 4]$ is computed for 1200 samples over a discrete grid of 100 points, i.e. mesh $8 \cdot 10^{-2}$, $N = 2^5, 2^6, \dots, 2^{16}$ for $\ell = 7$. The method tends to pick the largest bandwidth.

4. Proofs

4.1. Proof of Theorem 2.6

The proof of Theorem 2.6 relies on Proposition 4.1 below, which establishes a Bernstein inequality for the fluctuations of $\bar{\mu}_T^{N,h}(dx) - \bar{\mu}_T^h(dx)$. Recall that $\bar{\mu}_T^{N,h}$ denotes the empirical distribution of the interacting particle system at time T generated by (2.2), while $\bar{\mu}_T^h$ denotes the law of the random variable defined by the abstract Euler scheme (see Definition 2.3).

Proposition 4.1. *Work under Assumptions A1, A2 and A3. Let $N \geq 2$ and $Nh \leq \mathfrak{C}$. For any real-valued bounded function φ defined on \mathbb{R}^d and any $\varepsilon \geq 0$, we have:*

$$\mathbb{P}\left(\int_{\mathbb{R}^d} \varphi d(\bar{\mu}_T^{N,h} - \bar{\mu}_T^h) \geq \varepsilon\right) \leq \kappa_1 \exp\left(-\frac{\kappa_2 N \varepsilon^2}{|\varphi|_{L^2(\mu_T)}^2 + |\varphi|_\infty \varepsilon}\right), \quad (4.1)$$

where κ_1 depends on \mathfrak{C} and \mathfrak{M} , and κ_2 only on \mathfrak{M} , both specified explicitly in the proof.

The proof of Proposition 4.1 follows the strategy of Theorem 18 in [18]. We repeat the main steps in order to remain self-contained and highlight the important modifications we need in the context of Euler scheme approximations. A key ingredient of the proof of Proposition 4.1 is the comparison between the interacting particle system defined in (2.2) and an auxiliary system $(Y_t^{1,h}, \dots, Y_t^{N,h})_{t \in [0, T]}$ consisting of N independent copies of the abstract Euler scheme (2.4) (i.e., without interaction):

$$\begin{cases} dY_t^{n,h} = b(\underline{t}, Y_t^{n,h}, \mu_{\underline{t}})dt + \sigma(\underline{t}, Y_t^{n,h})dB_t^n, & 1 \leq n \leq N, \\ (Y_0^{1,h}, \dots, Y_0^{N,h}) = (\bar{X}_0^{1,h}, \dots, \bar{X}_0^{N,h}), \end{cases} \quad (4.2)$$

For $t \in [0, T]$, let

$$L_t^{N,h} := \sum_{n=1}^N \int_0^t \left((c^{-1/2}b)(\underline{s}, Y_{\underline{s}}^{n,h}, \frac{1}{N} \sum_{n=1}^N \delta_{Y_{\underline{s}}^{n,h}}) - (c^{-1/2}b)(\underline{s}, Y_{\underline{s}}^{n,h}, \mu_{\underline{s}}) \right)^\top dB_s^n, \quad (4.3)$$

where $c^{-1/2}$ is any square root of $c^{-1} = (\sigma\sigma^\top)^{-1}$, which is well defined by the ellipticity assumption A2. The next lemma provides a key estimate that enables a change of probability argument in the proof of Proposition 4.1. Its proof is postponed to the end of this section.

Lemma 4.2. *Work under Assumptions A1, A2 and A3. Assume that $Nh \leq \mathfrak{C}$ for some constant $\mathfrak{C} > 0$. For every $\kappa > 0$, there exists $\delta(\kappa) > 0$ such that*

$$\forall \delta \in [0, \delta(\kappa)], \quad \sup_{t \in [0, T-\delta(\kappa)]} \mathbb{E}_{\mathbb{P}} \left[\exp \left(\kappa \left(\langle L_t^{N,h} \rangle_{t+\delta} - \langle L_t^{N,h} \rangle_t \right) \right) \right] \leq C' \quad (4.4)$$

where $C' > 0$ only depends on \mathfrak{C} and \mathfrak{M} .

Proof of Proposition 4.1. We work on a rich enough filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ in order to accommodate all the random quantities needed. First, note that the abstract Euler scheme $(\bar{X}_t^h)_{t \in [0, T]}$ defined by (2.4) solves

$$d\bar{X}_t^h = b(\underline{t}, \bar{X}_t^h, \mu_{\underline{t}})dt + \sigma(\underline{t}, \bar{X}_t^h)dB_t, \quad \mathcal{L}(\bar{X}_0) = \mu_0, \quad (4.5)$$

where \underline{t} is defined by

$$\forall 0 \leq m \leq M \text{ and } t \in [t_m, t_{m+1}), \quad \underline{t} := t_m.$$

Similarly, $(\bar{X}_t^{1,h}, \dots, \bar{X}_t^{N,h})_{t \in [0, T]}$ defined by (2.2) solves

$$\begin{cases} d\bar{X}_t^{n,h} = b(\underline{t}, \bar{X}_t^{n,h}, \bar{\mu}_{\underline{t}}^{N,h})dt + \sigma(\underline{t}, \bar{X}_t^{n,h})dB_t^n, & 1 \leq n \leq N, \\ \bar{\mu}_{\underline{t}}^{N,h} = \frac{1}{N} \sum_{n=1}^N \delta_{\bar{X}_t^{n,h}}, \\ \mathcal{L}(\bar{X}_0^{1,h}, \dots, \bar{X}_0^{N,h}) = \mu_0^{\otimes N}. \end{cases} \quad (4.6)$$

Step 1. Consider the particle system $(Y_t^{1,h}, \dots, Y_t^{N,h})_{t \in [0, T]}$ defined in (4.2) and the process $(L_t^{N,h})_{t \in [0, T]}$ introduced in (4.3). For $t \in [0, T]$, define

$$\mathcal{E}_t(L^{N,h}) := \exp \left(L_t^{N,h} - \frac{1}{2} \langle L^{N,h} \rangle_t \right),$$

where $\langle L^{N,h} \rangle_t$ denotes the predictable compensator of $L_t^{N,h}$. Applying Lemma 4.2 with $\kappa = \frac{1}{2}$, together with Novikov's criterion (see, e.g., [29, Proposition 3.5.12, Corollary 3.5.13, and 3.5.14]), we deduce that $(\mathcal{E}_t(L^{N,h}))_{t \in [0, T]}$ is an $(\mathcal{F}_t, \mathbb{P})$ -martingale as soon as $Nh \leq \mathfrak{C}$. This allows us to define a new probability distribution \mathbb{Q} on (Ω, \mathcal{F}_T) by

$$\mathbb{Q} := \mathcal{E}_T(L^{N,h}) \cdot \mathbb{P}. \quad (4.7)$$

By Girsanov's Theorem (see e.g. [29, Theorem 3.5.1]), we have

$$\mathbb{Q} \circ (Y_t^{1,h}, \dots, Y_t^{N,h})_{t \in [0,T]}^{-1} = \mathbb{P} \circ (X_t^{1,h}, \dots, X_t^{N,h})_{t \in [0,T]}^{-1}. \quad (4.8)$$

Step 2. We claim that for any subdivision $0 = v_0 < v_1 < \dots < v_K = T$ and for any \mathcal{F}_T -measurable event A^N , we have

$$\mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_{j-1}} \right) \right] \leq \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \right]^{\frac{1}{4}} \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \right) \right]^{\frac{1}{4}}, \quad 1 \leq j \leq K. \quad (4.9)$$

The proof is the same as in Step 3 of the proof of Theorem 18 in [18] and is inspired from the estimate (4.2) in Theorem 2.6 in [32]. We repeat the argument: we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_{j-1}} \right) \right] &= \mathbb{E}_{\mathbb{P}} \left[\mathbb{E}_{\mathbb{Q}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \mid \mathcal{F}_{v_{j-1}} \right] \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[\mathbb{E}_{\mathbb{P}} \left[\frac{\mathcal{E}_{v_j}(L^{N,h})}{\mathcal{E}_{v_{j-1}}(L^{N,h})} \mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \mid \mathcal{F}_{v_{j-1}} \right] \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{\mathcal{E}_{v_j}(L^{N,h})}{\mathcal{E}_{v_{j-1}}(L^{N,h})} \mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \right], \end{aligned}$$

where the second inequality follows from Bayes's rule in [29, Lemma 3.5.3]. Next, we have

$$\frac{\mathcal{E}_{v_j}(L^{N,h})}{\mathcal{E}_{v_{j-1}}(L^{N,h})} = \mathcal{E}_{v_j} \left(2(L^{N,h} - L_{v_{j-1}}^{N,h}) \right)^{\frac{1}{2}} \cdot \exp \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right)^{\frac{1}{2}}.$$

The process $\left(\mathcal{E}_{v_j} \left(2(L^{N,h} - L_{v_{j-1}}^{N,h}) \right) \right)_{t \in [v_{j-1}, T]}$ is a $(\mathcal{F}_t, \mathbb{P})$ martingale if $Nh \leq \mathfrak{C}$. Hence

$$\mathbb{E}_{\mathbb{P}} \left[\mathcal{E}_{v_j} \left(2(L^{N,h} - L_{v_{j-1}}^{N,h}) \right) \right] = 1$$

for every $t \in [v_{j-1}, T]$. It follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_{j-1}} \right) \right] &= \mathbb{E}_{\mathbb{P}} \left[\mathcal{E}_{v_j} \left(2(L^{N,h} - L_{v_{j-1}}^{N,h}) \right)^{\frac{1}{2}} \cdot \exp \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right)^{\frac{1}{2}} \mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \right] \\ &\leq \left(\mathbb{E}_{\mathbb{P}} \left[\exp \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right)^2 \right] \right)^{\frac{1}{2}} \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \right) \right]^{\frac{1}{4}} \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right)^4 \right]^{\frac{1}{4}} \\ &\leq \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \right) \right]^{\frac{1}{4}} \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q} \left(A^N \mid \mathcal{F}_{v_j} \right) \right]^{\frac{1}{4}}, \end{aligned}$$

where the first two inequalities follows from Cauchy–Schwarz's inequality and the last inequality follows from Jensen's inequality. Thus (4.9) is established.

Step 3. Let $A^N \in \mathcal{F}_T$. Since $\left(\mathcal{E}_t(L^{N,h}) \right)_{t \in [0,T]}$ is a $(\mathcal{F}_t, \mathbb{P})$ martingale, \mathbb{P} and \mathbb{Q} coincide on \mathcal{F}_0 (see e.g. [29, Section 3, (5.5)]). It follows that

$$\mathbb{Q}(A^N) = \mathbb{E}_{\mathbb{Q}} \left[\mathbb{Q}(A^N \mid \mathcal{F}_0) \right] = \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q}(A^N \mid \mathcal{F}_0) \right].$$

Now, let $m \geq 1$ and take a subdivision $0 = v_0 < v_1 < \dots < v_m = T$ such that

$$\forall 0 \leq k \leq m-1, \quad v_{k+1} - v_k \leq \delta(2)$$

where $\delta(2)$ is the constant in Lemma 4.2 for $\kappa = 2$. It follows by (4.9) that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q}(A^N \mid \mathcal{F}_0) \right] &\leq \mathbb{E}_{\mathbb{P}} \left[\mathbb{Q}(A^N \mid \mathcal{F}_T) \right]^{4^{-m}} \prod_{j=1}^m \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \right) \right]^{\frac{j}{4}} \\ &\leq \mathbb{P}(A^N)^{4^{-m}} \prod_{j=1}^m \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{v_j} - \langle L^{N,h} \rangle_{v_{j-1}} \right) \right) \right]^{\frac{j}{4}} \quad (\text{since } A^N \in \mathcal{F}_T) \end{aligned}$$

$$\begin{aligned}
 &\leq \mathbb{P}(A^N)^{4^{-m}} \left(\sup_{t \in [0, T-\delta(2)]} \mathbb{E}_{\mathbb{P}} \left[\exp \left(2 \left(\langle L^{N,h} \rangle_{t+\varepsilon} - \langle L^{N,h} \rangle_t \right) \right) \right] \right)^{\frac{m(m+1)}{8}} \\
 &\leq \mathbb{P}(A^N)^{4^{-m}} C' \frac{m(m+1)}{8}
 \end{aligned} \tag{4.10}$$

by applying Lemma 4.2 with $\kappa = 2$.

Step 4. We first recall the Bernstein's inequality: if Z_1, \dots, Z_N are real-valued centered independent random variables bounded by some constant Q , we have

$$\forall \varepsilon \geq 0, \quad \mathbb{P} \left(\sum_{n=1}^N Z_n \geq \varepsilon \right) \leq \exp \left(- \frac{\varepsilon^2}{2 \sum_{n=1}^N \mathbb{E}[Z_n^2] + \frac{2}{3} Q \varepsilon} \right). \tag{4.11}$$

Now, for $\varepsilon \geq 0$, we pick

$$A_T^N := \left\{ \sum_{n=1}^N \left(\varphi(Y_T^{n,h}) - \int_{\mathbb{R}^d} \varphi(y) \bar{\mu}_T^h(dy) \right) \geq N\varepsilon \right\}$$

so that $A_T^N \in \mathcal{F}_T$. Since $(Y_T^{1,h}, \dots, Y_T^{N,h})$ are independent with common distribution $\bar{\mu}_T^h$, we have

$$\mathbb{P}(A_T^N) \leq \exp \left(- \frac{N\varepsilon^2}{2|\varphi|_{L^2(\bar{\mu}_T^h)}^2 + \frac{2}{3}|\varphi|_{\infty}\varepsilon} \right). \tag{4.12}$$

by (4.11) with $Z_n = \varphi(Y_T^{n,h}) - \int_{\mathbb{R}^d} \varphi(x) \bar{\mu}_T^h(dx)$, having $\mathbb{E}[Z_n^2] \leq |\varphi|_{L^2(\bar{\mu}_T^h)}^2$ and $Q = |\varphi|_{\infty}$. It follows by (4.8) that

$$\mathbb{P} \left(\int_{\mathbb{R}^d} \varphi d(\bar{\mu}_T^{N,h} - \bar{\mu}_T^h) \geq \varepsilon \right) = \mathbb{P} \left(\sum_{n=1}^N \left(\varphi(\bar{X}_T^{n,h}) - \int_{\mathbb{R}} \varphi(y) \bar{\mu}_T^h(dy) \right) \geq N\varepsilon \right) = \mathbb{Q}(A_T^N).$$

By Step 3, we infer

$$\mathbb{Q}(A_T^N) \leq \mathbb{P}(A_T^N)^{4^{-m}} C' \frac{m(m+1)}{8} \leq C' \frac{m(m+1)}{8} \exp \left(-4^{-m} \frac{N\varepsilon^2}{2|\varphi|_{L^2(\bar{\mu}_T^h)}^2 + \frac{2}{3}|\varphi|_{\infty}\varepsilon} \right),$$

and we conclude by taking $\kappa_1 = C' \frac{m(m+1)}{8}$ and $\kappa_2 = 2^{-1}4^{-m}$. ■

Completion of proof of Theorem 2.6. Recall the notation $K_{\eta}(x) = \eta^{-d}K(\eta^{-1}x)$ and set $\varphi \star \psi(x) = \int_{\mathbb{R}^d} \varphi(x-y)\psi(y)dy$ for the convolution product between two integrable functions $\varphi, \psi : \mathbb{R}^d \rightarrow \mathbb{R}$. We have

$$\begin{aligned}
 &\mathbb{P} \left(|\hat{\mu}_T^{N,h,\eta}(x) - \mu_T(x)| \geq \varepsilon \right) \\
 &\leq \mathbb{P} \left(|\hat{\mu}_T^{N,h,\eta}(x) - K_{\eta} \star \bar{\mu}_T^h(x)| \geq \varepsilon/3 \right) + \mathbf{1}_{\{|K_{\eta} \star \bar{\mu}_T^h(x) - K_{\eta} \star \mu_T(x)| \geq \varepsilon/3\}} + \mathbf{1}_{\{|K_{\eta} \star \mu_T(x) - \mu_T(x)| \geq \varepsilon/3\}}. \tag{4.13}
 \end{aligned}$$

We have

$$|K_{\eta} \star \bar{\mu}_T^h(x) - K_{\eta} \star \mu_T(x)| \leq |K|_{L^1} \sup_{x \in \mathbb{R}^d} |\bar{\mu}_T^h(x) - \mu_T(x)|$$

since $|K_{\eta}|_{L^1} = |K|_{L^1}$ and this last term is strictly smaller than $\varepsilon/3$ for $h < h^* \left(|K|_{L^1}^{-1} \frac{1}{3} \varepsilon \right)$, therefore the second indicator is 0. Likewise

$$|K_{\eta} \star \mu_T(x) - \mu_T(x)| \leq \mathcal{B}_{\eta}(\mu_T, x) < \varepsilon/3$$

for $\eta < \eta^*(\frac{1}{3}\varepsilon, \mu_T)$ and the third indicator is 0 as well. Finally

$$\begin{aligned} & \mathbb{P}\left(\left|\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x)\right| \geq \frac{1}{3}\varepsilon\right) \\ &= \mathbb{P}\left(\left|\int_{\mathbb{R}^d} K_\eta(x - \cdot) d(\bar{\mu}_T^{N,h} - \bar{\mu}_T^h)\right| \geq \frac{1}{3}\varepsilon\right) \leq 2\kappa_1 \exp\left(-\frac{\kappa_2 N (\frac{1}{3}\varepsilon)^2}{|K_\eta(x - \cdot)|_{L^2(\mu_T)}^2 + |K_\eta|_\infty \varepsilon}\right) \end{aligned}$$

by Proposition 4.1. Theorem 2.6 then follows, where we reuse κ_2 (instead of $\kappa_2/9$) for simplicity, as constants may be redefined without loss of generality. \blacksquare

Proof of Lemma 4.2. Writing $\tilde{\mu}_s^{N,h} = \frac{1}{N} \sum_{n=1}^N \delta_{Y_s^{n,h}}$, for $\kappa > 0$, we have

$$\begin{aligned} & \kappa \left(\langle L^{N,h} \rangle_{t+\delta} - \langle L^{N,h} \rangle_t \right) \\ &= \kappa \sum_{n=1}^N \int_t^{t+\delta} \left(b(\underline{s}, Y_s^{n,h}, \tilde{\mu}_s^{N,h}) - b(\underline{s}, Y_s^{n,h}, \mu_s) \right)^\top c(\underline{s}, Y_s^{n,h})^{-1} \left(b(\underline{s}, Y_s^{n,h}, \tilde{\mu}_s^{N,h}) - b(\underline{s}, Y_s^{n,h}, \mu_s) \right) ds \\ &\leq \kappa \sup_{t \in [0, T]} |\text{Tr}(c(t, \cdot)^{-1})|_\infty \sum_{n=1}^N \int_t^{t+\delta} |b(\underline{s}, Y_s^{n,h}, \tilde{\mu}_s^{N,h}) - b(\underline{s}, Y_s^{n,h}, \mu_s)|^2 ds \\ &\leq 2\kappa \sup_{t \in [0, T]} |\text{Tr}(c(t, \cdot)^{-1})|_\infty \\ &\quad \times \left(\sum_{n=1}^N \int_t^{t+\delta} |b(\underline{s}, Y_s^{n,h}, \tilde{\mu}_s^{N,h}) - b(\underline{s}, Y_s^{n,h}, \bar{\mu}_s^h)|^2 ds + N |\tilde{b}|_{\text{Lip}}^2 \int_t^{t+\delta} \mathcal{W}_1(\bar{\mu}_s^h, \mu_s)^2 ds \right), \end{aligned} \quad (4.14)$$

by the uniform ellipticity of c thanks to Assumption A2 and by triangle inequality for the last inequality, with $|\tilde{b}(t, x, y) - \tilde{b}(t, x, y')| \leq |\tilde{b}|_{\text{Lip}} |y - y'|$ in particular. Recall that under Assumption A1, A2 and A3, there exists a unique strong solution $X = (X_t)_{t \in [0, T]}$ of the McKean–Vlasov equation (1.2) satisfying

$$\forall s, t \in [0, T], s < t, \quad \mathbb{E} \left[|X_t - X_s|^2 \right]^{1/2} \leq \mathfrak{M} \sqrt{t - s},$$

see [38, Proposition 2.1]. Hence, the function $(t, x) \mapsto b^\mu(t, x) := b(t, x, \mu_t)$ with $\mu_t = \mathcal{L}(X_t)$ is $\frac{1}{2}$ -Hölder continuous in t and Lipschitz continuous in x . Thus, one can obtain

$$\sup_{s \in [0, T]} \mathcal{W}_1(\bar{\mu}_s^h, \mu_s) \leq \mathfrak{M} h^{1/2},$$

by rewriting (1.2) as a Brownian diffusion

$$dX_t = b^\mu(t, X_t) dt + \sigma(t, X_t) dB_t$$

and by applying the classical convergence result of the Euler scheme for a Brownian diffusion (see e.g. [46, Theorem 7.2]) since for every $s \in [0, T]$, $\mathcal{W}_1(\bar{\mu}_s^h, \mu_s) \leq \mathbb{E}[|X_s - \bar{X}_s^h|]$. Writing

$$b(\underline{s}, Y_s^{n,h}, \tilde{\mu}_s^{N,h}) - b(\underline{s}, Y_s^{n,h}, \bar{\mu}_s^h) = N^{-1} \sum_{n'=1}^N \xi_s^{n,n'} = N^{-1} \xi_s^{n,n} + \frac{N-1}{N} S_s^{n,N},$$

with $S_s^{n,N} = \frac{1}{N-1} \sum_{n' \neq n} \xi_s^{n,n'}$ and

$$\xi_s^{n,n'} = \tilde{b}(\underline{s}, Y_s^{n,h}, Y_s^{n',h}) - \int_{\mathbb{R}^d} \tilde{b}(\underline{s}, Y_s^{n,h}, y) \bar{\mu}_s^h(dy).$$

The random variables $(\xi_s^{n,n'})_{1 \leq n' \leq N, n' \neq n}$ are centered, identically distributed and conditionally independent given $Y_s^{n,h}$. Moreover, for every integer $m \geq 1$, we have the estimate

$$\sup_{s \in [0, T]} \mathbb{E} \left[|Y_s^{n,h}|^{2m} \right] \leq \mathfrak{M}^m m!. \quad (4.15)$$

The proof of (4.15) is standard (see for instance, [43, 51] or [18] for a control of growth in the constant m and we omit it). From (4.15), we infer, for every $m \geq 1$:

$$\begin{aligned} \mathbb{E}\left[|\xi_s^{n,n'}|^{2m} \mid Y_{\underline{s}}^{n,h}\right] &= \mathbb{E}\left[\left|\tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, Y_{\underline{s}}^{n',h}) - \int_{\mathbb{R}^d} \tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, y) \bar{\mu}_{\underline{s}}^h(dy)\right|^{2m} \mid Y_{\underline{s}}^{n,h}\right] \\ &\leq \mathbb{E}\left[2^{2m-1} \left(\left|\tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, Y_{\underline{s}}^{n',h}) - \tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, 0)\right|^{2m}\right.\right. \\ &\quad \left.\left.+ \left|\tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, 0) - \int_{\mathbb{R}^d} \tilde{b}(\underline{s}, Y_{\underline{s}}^{n,h}, y) \bar{\mu}_{\underline{s}}^h(dy)\right|^{2m}\right) \mid Y_{\underline{s}}^{n,h}\right] \\ &\leq \mathbb{E}\left[2^{2m-1} \left(\tilde{b}_{\text{Lip}}^{2m} |Y_{\underline{s}}^{n',h}|^{2m} + \tilde{b}_{\text{Lip}}^{2m} \mathbb{E}\left[|Y_{\underline{s}}^{n',h}|^{2m}\right]\right) \mid Y_{\underline{s}}^{n,h}\right] = 2^{2m} \tilde{b}_{\text{Lip}}^{2m} \mathbb{E}\left[|Y_{\underline{s}}^{n',h}|^{2m}\right] \\ &\leq \left(4\tilde{b}_{\text{Lip}}^2 \mathfrak{M}\right)^m m! \end{aligned}$$

and in turn, we derive, for every $v \in \mathbb{R}$,

$$\mathbb{E}\left[\exp(v(\xi_s^{n,n'})_{[k]}) \mid Y_{\underline{s}}^{n,h}\right] \leq \exp\left(\frac{\mathfrak{M}v^2}{2}\right), \quad k = 1, \dots, d.$$

In other words, conditional on $Y_{\underline{s}}^{n,h}$, each component $(\xi_s^{n,n'})_{[k]}$ of $\xi_s^{n,n'} = ((\xi_s^{n,n'})_{[1]}, \dots, (\xi_s^{n,n'})_{[d]})$ is sub-Gaussian with variance proxy depending only on the model constant \mathfrak{M} , which we continue to denote by \mathfrak{M} in the subsequent computations (see e.g. [12] and [47, Theorem 2.1.1]). Consequently, conditional on $Y_{\underline{s}}^{n,h}$, each component $(S_s^{n,N})_{[k]}$ of $S_s^{n,N}$ is sub-Gaussian with variance proxy $(N-1)^{-1}\mathfrak{M}$, by the conditional independence of the random variables $(\xi_s^{n,n'})_{1 \leq n' \leq N, n' \neq n}$. This implies in particular

$$\mathbb{E}\left[\exp\left(\frac{(N-1)(S_s^{n,N})_{[k]}^2}{8\mathfrak{M}}\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(\frac{(N-1)(S_s^{n,N})_{[k]}^2}{8\mathfrak{M}}\right) \mid Y_{\underline{s}}^{n,h}\right]\right] \leq 2,$$

and in turn

$$\mathbb{E}\left[\exp\left(\frac{(N-1)|S_s^{n,N}|^2}{8d\mathfrak{M}}\right)\right] \leq d^{-1} \sum_{k=1}^d \mathbb{E}\left[\exp\left(\frac{(N-1)(S_s^{n,N})_{[k]}^2}{8\mathfrak{M}}\right)\right] \leq 2. \quad (4.16)$$

Likewise

$$\mathbb{E}\left[\exp\left(\frac{|\xi_s^{n,n}|^2}{8d\mathfrak{M}}\right)\right] \leq 2. \quad (4.17)$$

Abbreviating $\tau = 2\kappa \sup_{t \in [0, T]} |\text{Tr}(c(t, \cdot)^{-1})|_{\infty}$, it follows that

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\kappa(\langle L^{N,h} \rangle_{t+\delta} - \langle L^{N,h} \rangle_t)\right)\right] \\ &\leq e^{\tau \tilde{b}_{\text{Lip}}^2 \mathfrak{M}^2 \delta N h} \mathbb{E}\left[\exp\left(\tau \int_t^{t+\delta} \sum_{n=1}^N |N^{-1} \sum_{n'=1}^N \xi_s^{n,n'}|^2 ds\right)\right] \\ &\leq e^{\tau \tilde{b}_{\text{Lip}}^2 \mathfrak{M}^2 \delta N h} \frac{1}{2\delta} \int_t^{t+\delta} N^{-1} \sum_{n=1}^N \left(\mathbb{E}\left[\exp(4\delta\tau N |N^{-1} \xi_s^{n,n}|^2)\right] + \mathbb{E}\left[\exp(4\delta\tau N |\frac{N-1}{N} S_s^{n,N}|^2)\right]\right) ds \\ &\leq 2e^{\tau \tilde{b}_{\text{Lip}}^2 \mathfrak{M}^2 \delta N h} \end{aligned}$$

as soon as $\delta \leq (32\tau d\mathfrak{M})^{-1} \frac{N}{N-1} = (64\kappa \sup_{t \in [0, T]} |\text{Tr}(c(t, \cdot)^{-1})|_{\infty} d\mathfrak{M})^{-1} \frac{N}{N-1}$ by (4.16) and (4.17). We obtain Lemma 4.2 with a constant C' depending only on \mathfrak{M} and \mathfrak{C} . \blacksquare

4.2. Proof of Theorem 2.7

The proof of Theorem 2.7 relies on a crucial approximation result, stated in Proposition 4.3 below, concerning the density of a diffusion process and that of its Euler scheme counterpart. The proof of Proposition 4.3 itself is based on the sharp results of Gobet and Labart [22].

Proposition 4.3. *Work under Assumptions A1, A2 and A3. Let $x \mapsto \bar{\mu}_t^h(x)$ be the density function of \bar{X}_t^h defined by (2.4). For any $0 < t_{\min} \leq T$, we have*

$$\sup_{(t,x) \in [t_{\min}, T] \times \mathbb{R}^d} |\bar{\mu}_t^h(x) - \mu_t(x)| \leq C_{t_{\min}, \mathfrak{M}} \cdot h, \quad (4.18)$$

for some constant $C_{t_{\min}, \mathfrak{M}}$ depending only on t_{\min} and \mathfrak{M} .

Proof of Proposition 4.3. For $x_0 \in \mathbb{R}^d$, let $(\xi_t^{x_0})_{t \in [0, T]}$ be a diffusion process of the form

$$\xi_t^{x_0} = x_0 + \int_0^t \tilde{b}(s, \xi_s^{x_0}) ds + \int_0^t \tilde{\sigma}(s, \xi_s^{x_0}) dB_s, \quad (4.19)$$

where $\tilde{\sigma} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ and $\tilde{b} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are both $\mathcal{C}_b^{1,3}$ and $\tilde{\sigma}$ is uniformly elliptic and $\partial_t \tilde{\sigma}$ is $\mathcal{C}_b^{0,1}$. We associate its companion Euler scheme $(\bar{\xi}_t^{x_0, h})_{t \in [0, T]}$:

$$\begin{cases} \bar{\xi}_{t_{m+1}}^{x_0, h} = \bar{\xi}_{t_m}^{x_0, h} + h \cdot \tilde{b}(t_m, \bar{\xi}_{t_m}^{x_0, h}) + \sqrt{h} \tilde{\sigma}(t_m, \bar{\xi}_{t_m}^{x_0, h}) Z_{m+1}^n, \\ Z_{m+1}^n := \frac{1}{\sqrt{h}} (B_{t_{m+1}} - B_{t_m}), \\ \bar{\xi}_{t_0}^{x_0, h} = x_0, \\ \forall t \in [t_m, t_{m+1}), \quad \bar{\xi}_t^{x_0, h} = \bar{\xi}_{t_m}^{x_0, h} + (t - t_m) \tilde{b}(t_m, \bar{\xi}_{t_m}^{x_0, h}) + \tilde{\sigma}(t_m, \bar{\xi}_{t_m}^{x_0, h}) (B_t - B_{t_m}). \end{cases}$$

For $t > 0$, both $\mathcal{L}(\xi_t^{x_0})$ and $\mathcal{L}(\bar{\xi}_t^{x_0, h})$ are absolutely continuous, with density $p_t(x_0, x)$ and $\bar{p}_t^h(x_0, x)$ w.r.t. the Lebesgue measure dx on \mathbb{R}^d . By Theorem 2.3 in [22], there exist two constants c_1 and c_2 depending on T , d and the data $(\tilde{\sigma}, \tilde{b})$ such that

$$|\bar{p}_t^h(x_0, x) - p_t(x_0, x)| \leq c_1 h t^{-\frac{d+1}{2}} \exp\left(-\frac{c_2 |x - x_0|^2}{t}\right). \quad (4.20)$$

Taking $\tilde{\sigma}(t, x) = \sigma(t, x)$ and $\tilde{b}(t, x) = b(t, x, \mu_t)$, we can identify $\bar{\mu}_t^h(x)$ with $\int_{\mathbb{R}^d} \bar{p}_t^h(x_0, x) \mu_0(dx_0)$ and $\mu_t(x)$ with $\int_{\mathbb{R}^d} p_t(x_0, x) \mu_0(dx_0)$. For every $t \in [t_{\min}, T]$, we have

$$\begin{aligned} |\bar{\mu}_t^h(x) - \mu_t(x)| &\leq \int_{\mathbb{R}^d} |\bar{p}_t^h(x_0, x) - p_t(x_0, x)| \mu_0(dx_0) \\ &\leq c_1 h \int_{\mathbb{R}^d} t^{-\frac{d+1}{2}} \exp\left(-\frac{c_2 |x - x_0|^2}{t}\right) \mu_0(dx_0) \\ &\leq c_1 t_{\min}^{-\frac{d+1}{2}} \cdot h =: C_{t_{\min}, \mathfrak{M}} \cdot h, \end{aligned}$$

by Theorem 2.3 in [22] since Assumption A3 implies $(t, x) \mapsto b(t, x, \mu_t)$ is $\mathcal{C}_b^{1,3}$. More specifically, for $(X_t)_{t \in [0, T]}$ solving (1.2), we have

$$b(t, x, \mu_t) = \mathbb{E}[\tilde{b}(t, x, X_t)].$$

By dominated convergence, $x \mapsto b(t, x, \mu_t)$ is \mathcal{C}_b^3 thanks to the regularity assumptions on $x \mapsto \tilde{b}(t, x, y)$. For the regularity in time, by Itô's formula

$$\mathbb{E}[\tilde{b}(t, \cdot, X_t)] = \mathbb{E}[\tilde{b}(0, \cdot, X_0)] + \int_0^t \mathbb{E}[\partial_1 \tilde{b}(s, \cdot, X_s)] ds + \int_0^t \mathbb{E}[\mathcal{A}_s \tilde{b}(s, \cdot, X_s)] ds,$$

where $(\mathcal{A}_s)_{s \in [0, T]}$ is the family of generators defined in (2.13) applied to

$$y \mapsto \tilde{b}(s, \cdot, y) = (\tilde{b}(s, \cdot, y)_{[1]}, \dots, \tilde{b}(s, \cdot, y)_{[d]})$$

componentwise. Moreover, for each component $1 \leq k \leq d$,

$$\mathcal{A}_s \tilde{b}(s, \cdot, X_s)_{[k]} = \mathbb{E}[\tilde{b}(s, \cdot, X_s)]^\top \nabla_y \tilde{b}(s, \cdot, X_s)_{[k]} + \frac{1}{2} \sum_{l, l'=1}^d c_{ll'}(s, X_s) \partial_{y_l y_{l'}} \tilde{b}(s, \cdot, X_s)_{[k]},$$

hence

$$\mathbb{E}[\mathcal{A}_s \tilde{b}(s, \cdot, X_s)_{[k]}] = \mathbb{E}[\tilde{b}(s, \cdot, X_s)]^\top \mathbb{E}[F_k(s, X_s)] + \mathbb{E}[G_k(s, X_s)],$$

with

$$F_k(s, y) = \nabla_y \tilde{b}(s, \cdot, y)_{[k]}, \quad \text{and} \quad G_k(s, y) = \frac{1}{2} \sum_{l, l'=1}^d c_{ll'}(s, y) \partial_{y_l y_{l'}} \tilde{b}(s, \cdot, y)_{[k]}.$$

All three functions $(s, y) \mapsto \tilde{b}(s, \cdot, y)$, $(s, y) \mapsto F_k(s, y)$ and $(s, y) \mapsto G_k(s, y)$ are continuous and bounded by Assumption A3, and so are $s \mapsto \mathbb{E}[\tilde{b}(s, \cdot, X_s)]$, $s \mapsto \mathbb{E}[F_k(s, X_s)]$ and $s \mapsto \mathbb{E}[G_k(s, X_s)]$ by dominated convergence using in particular that $s \mapsto X_s$ is stochastically continuous. Hence $s \mapsto \mathbb{E}[\mathcal{A}_s \tilde{b}(s, \cdot, X_s)]$ is continuous and $s \mapsto \mathbb{E}[\partial_s \tilde{b}(s, \cdot, X_s)]$ too, using again Assumption A3. It follows that $t \mapsto b(t, x, \mu_t)$ is \mathcal{C}_b^1 on $[0, T]$. \blacksquare

Proof of Theorem 2.7. We have

$$\begin{aligned} & \mathbb{E}[|\widehat{\mu}_T^{N, h, \eta}(x) - \mu_T(x)|^p] \\ & \leq 3^{p-1} \left(\mathbb{E}[|\widehat{\mu}_T^{N, h, \eta}(x) - K_\eta \star \bar{\mu}_T^h(x)|^p] + |K_\eta \star \bar{\mu}_T^h(x) - K_\eta \star \mu_T(x)|^p + |K_\eta \star \mu_T(x) - \mu_T(x)|^p \right), \end{aligned}$$

and we plan to bound each term separately. First, by Proposition 4.1

$$\begin{aligned} \mathbb{E}[|\widehat{\mu}_T^{N, h, \eta}(x) - K_\eta \star \bar{\mu}_T^h(x)|^p] &= \int_0^\infty \mathbb{P}\left(|\widehat{\mu}_T^{N, h, \eta}(x) - K_\eta \star \bar{\mu}_T^h(x)| \geq z^{1/p}\right) dz \\ &\leq 2\kappa_1 \int_0^\infty \exp\left(-\frac{\kappa_2 N z^{2/p}}{|K_\eta(x - \cdot)|_{L^2(\mu_T)}^2 + |K_\eta|_\infty z^{1/p}}\right) dz \\ &\leq 2\kappa_1 \int_0^\infty \exp\left(-\frac{\kappa_2 N \eta^d z^{2/p}}{\sup_{y \in \text{Supp}(K)} \mu_T(x - y) |K|_{L^2}^2 + |K|_\infty z^{1/p}}\right) dz \\ &\leq 2\kappa_1 c_p \kappa_2^{-p/2} \sup_{y \in \text{Supp}(K)} \mu_T(x - y)^{p/2} |K|_{L^2}^p (N \eta^d)^{-p/2}, \end{aligned}$$

stemming from the estimate

$$\int_0^\infty \exp\left(-\frac{az^{2/p}}{b + cz^{1/p}}\right) dz \leq c_p \max\left(\left(\frac{a}{b}\right)^{-p/2}, \left(\frac{a}{c}\right)^{-p}\right),$$

valid for $a, b, c, p > 0$, with $c_p = 2 \int_0^\infty \exp\left(-\frac{1}{2}(\min(z, \sqrt{z}))^{2/p}\right) dz$. Next,

$$|K_\eta \star \bar{\mu}_T^h(x) - K_\eta \star \mu_T(x)|^p \leq \sup_{y \in \mathbb{R}^d} |\bar{\mu}_T^h(y) - \mu_T(y)|^p \left(\int_{\mathbb{R}^d} |K_\eta(x - y)| dy\right)^p \leq \mathfrak{M}^p |K|_{L^1}^p h^p$$

by Proposition 4.3. Finally, by definition

$$|K_\eta \star \mu_T(x) - \mu_T(x)|^p \leq \mathcal{B}_\eta(\mu_T, x)^p,$$

and we obtain Theorem 2.7 with

$$\kappa_3 = 3^{p-1} \max\left(2\kappa_1 c_p \kappa_2^{-p/2} \sup_{x \in \mathcal{K}, y \in \text{Supp}(K)} \mu_T(x - y)^{p/2} |K|_{L^2}^p, \mathfrak{M}^p |K|_{L^1}^p, 1\right). \quad \blacksquare$$

Proof of Corollary 2.8. By Taylor's formula, we have, for $x, y \in \mathbb{R}^d$, $\eta > 0$ and any $1 \leq k' \leq k$:

$$\mu_T(x + \eta y) - \mu_T(x) = \sum_{1 \leq |\alpha| \leq k'-1} \frac{\partial^\alpha \mu_T(x)}{\alpha!} (\eta y)^\alpha + k' \sum_{|\alpha|=k'} \frac{(\eta y)^\alpha}{\alpha!} \int_0^1 (1-t)^{k'-1} \partial^\alpha \mu_T(x + t\eta y) dt, \quad (4.21)$$

with multi-index notation $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, \dots\}$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, and for $x = (x_1, \dots, x_d)$:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}, \quad \text{and} \quad \partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_d} x_d}.$$

It follows that

$$\begin{aligned} & K_\eta \star \mu_T(x) - \mu_T(x) \\ &= \eta^{-d} \int_{\mathbb{R}^d} K(\eta^{-1}(x-y)) (\mu_T(y) - \mu_T(x)) dy \\ &= \int_{\mathbb{R}^d} K(-y) (\mu_T(x + \eta y) - \mu_T(x)) dy \\ &= \int_{\mathbb{R}^d} K(-y) \left(\min(k, \ell+1) \sum_{|\alpha|=\min(k, \ell+1)} \frac{(\eta y)^\alpha}{\alpha!} \int_0^1 (1-t)^{\min(k, \ell+1)-1} \partial^\alpha \mu_T(x + t\eta y) dt \right) dy, \end{aligned}$$

thanks to (4.21) with $k' = \min(k, \ell+1)$ and the cancellation property (2.3) of the kernel K that eliminates the polynomial term in the Taylor's expansion, hence

$$|K_\eta \star \mu_T(x) - \mu_T(x)| \leq |\mu_T|_{\min(k, \ell+1), K} \eta^{\min(k, \ell+1)},$$

with

$$|f|_{\min(k, \ell+1), K} = \sum_{|\alpha|=\min(k, \ell+1)} \frac{|\partial^\alpha f|_\infty}{\alpha!} \int_{\mathbb{R}^d} |y|^\alpha |K(y)| dy. \quad (4.22)$$

It follows that

$$\mathcal{B}_\eta(\mu_T, x)^p \leq |\mu_T|_{\min(k, \ell+1), K}^p \eta^{\min(k, \ell+1)p}. \quad (4.23)$$

Applying now Theorem 2.7, we obtain

$$\begin{aligned} \mathbb{E} \left[|\widehat{\mu}_T^{N, h, \eta}(x) - \mu_T(x)|^p \right] &\leq \kappa_3 \left(|\mu_T|_{\min(k, \ell+1), K}^p \eta^{\min(k, \ell+1)p} + (N\eta^d)^{-p/2} + h^p \right) \\ &\leq \kappa_3 \left(|\mu_T|_{\min(k, \ell+1), K}^p + 1 \right) N^{-\min(k, \ell+1)p/(2\min(k, \ell+1)+1)} + h^p \end{aligned}$$

with the choice $\eta = N^{-1/(2\min(k, \ell+1)+d)}$ hence the corollary with $\kappa_4 = \kappa_3(1 + |\mu_T|_{\min(k, \ell+1), K}^p)$. \blacksquare

4.3. Proof of Theorem 2.9

This is an adaptation of the classical Goldenshluger–Lepski method [23, 24, 25]. We repeat the main arguments and highlight the differences due to the stochastic approximation and the presence of the Euler scheme.

Abbreviating $\mathbf{A}_\eta^N(T, x)$ by \mathbf{A}_η^N , we have, for every $\eta \in \mathcal{H}$,

$$\begin{aligned}
 & \mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \mu_T(x) \right)^2 \right] \\
 & \leq 2\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \widehat{\mu}_T^{N,h,\eta}(x) \right)^2 \right] + 2\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x) \right)^2 \right] \\
 & \leq 2\mathbb{E} \left[\left\{ \left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \widehat{\mu}_T^{N,h,\eta}(x) \right)^2 - \mathbf{V}_\eta^N - \mathbf{V}_{\widehat{\eta}^N(T,x)}^N \right\}_+ + \mathbf{V}_\eta^N + \mathbf{V}_{\widehat{\eta}^N(T,x)}^N \right] \\
 & \quad + 2\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x) \right)^2 \right] \\
 & \leq 2\mathbb{E} \left[\mathbf{A}_{\max(\eta, \widehat{\eta}^N(T,x))}^N + \mathbf{V}_\eta^N + \mathbf{V}_{\widehat{\eta}^N(T,x)}^N \right] + 2\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x) \right)^2 \right] \\
 & \leq 2 \left(\mathbb{E} \left[\mathbf{A}_\eta^N \right] + \mathbf{V}_\eta^N \right) + 2\mathbb{E} \left[\mathbf{A}_{\widehat{\eta}^N(T,x)}^N + \mathbf{V}_{\widehat{\eta}^N(T,x)}^N \right] + 2\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\eta}(x) - \mu_T(x) \right)^2 \right] \\
 & \leq 4\mathbb{E} \left[\mathbf{A}_\eta^N + \mathbf{V}_\eta^N \right] + \kappa_3 (\mathcal{B}_\eta(x, \mu_T))^2 + (N\eta^d)^{-1} + h^2 \\
 & \leq \kappa_7 \left(\mathbb{E} \left[\mathbf{A}_\eta^N \right] + \mathbf{V}_\eta^N + \mathcal{B}_\eta(x, \mu_T)^2 + h^2 \right)
 \end{aligned}$$

with $\kappa_7 = 2 \max(2 + \frac{\kappa_3}{\varpi|K|_{L^2}^2}, \kappa_3)$ by definition of $\widehat{\eta}^N(T, x)$ in (2.11), of \mathbf{V}_η^N in (2.10) and using Theorem 2.7 with $p = 2$ to bound the last term.

We next bound the term $\mathbb{E} \left[\mathbf{A}_\eta^N \right]$. For $\eta, \eta' \in \mathcal{H}$, with $\eta' \leq \eta$, we start with the decomposition

$$\begin{aligned}
 & \widehat{\mu}_T^{N,h,\eta}(x) - \widehat{\mu}_T^{N,h,\eta'}(x) \\
 & = \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right) + \left(K_\eta \star \bar{\mu}_T^h(x) - K_\eta \star \mu_T(x) \right) + \left(K_\eta \star \mu_T(x) - \mu_T(x) \right) \\
 & \quad - \left(\widehat{\mu}_T^{N,h,\eta'}(x) - K_{\eta'} \star \bar{\mu}_T^h(x) \right) - \left(K_{\eta'} \star \bar{\mu}_T^h(x) - K_{\eta'} \star \mu_T(x) \right) - \left(K_{\eta'} \star \mu_T(x) - \mu_T(x) \right),
 \end{aligned}$$

and thus, by Proposition 4.3, we infer

$$\begin{aligned}
 & \left| \widehat{\mu}_T^{N,h,\eta}(x) - \widehat{\mu}_T^{N,h,\eta'}(x) \right| \\
 & \leq \left| \widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right| + \left| \widehat{\mu}_T^{N,h,\eta'}(x) - K_{\eta'} \star \bar{\mu}_T^h(x) \right| + 2\mathfrak{M}|K|_{L^1}h + 2\mathcal{B}_\eta(\mu_T, x)
 \end{aligned}$$

using $\eta' \leq \eta$ to bound the second bias term.

$$\begin{aligned}
 & \left(\widehat{\mu}_T^{N,h,\eta}(x) - \widehat{\mu}_T^{N,h,\eta'}(x) \right)^2 - \mathbf{V}_\eta^N - \mathbf{V}_{\eta'}^N \\
 & \leq 3 \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_\eta^N + 3 \left(\widehat{\mu}_T^{N,h,\eta'}(x) - K_{\eta'} \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_{\eta'}^N \\
 & \quad + 3(2\mathfrak{M}|K|_{L^1}h + 2\mathcal{B}_\eta(x, \mu_T))^2.
 \end{aligned}$$

Taking maximum for $\eta' \leq \eta$, we thus obtain

$$\begin{aligned}
 & \max_{\eta' \leq \eta} \left\{ \left(\widehat{\mu}_T^{N,h,\eta}(x) - \widehat{\mu}_T^{N,h,\eta'}(x) \right)^2 - \mathbf{V}_\eta^N - \mathbf{V}_{\eta'}^N \right\} \\
 & \leq \left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \mu_T(x) \right)^2 - \mathbf{V}_\eta^N \right\}_+ + \max_{\eta' \leq \eta} \left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta'}(x) - K_{\eta'} \star \mu_T(x) \right)^2 - \mathbf{V}_{\eta'}^N \right\}_+ \\
 & \quad + 24 \left(\mathfrak{M}^2 |K|_{L^1}^2 h^2 + \mathcal{B}_\eta(x, \mu_T)^2 \right).
 \end{aligned}$$

We finally bound the expectation of each term. First, we have, by Proposition 4.1

$$\begin{aligned}
& \mathbb{E} \left[\left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_\eta^N \right\}_+ \right] \\
&= \int_0^\infty \mathbb{P} \left(3 \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_\eta^N \geq z \right) dz \\
&= \int_0^\infty \mathbb{P} \left(\left| \widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right| \geq 3^{-1/2} (z + \mathbf{V}_\eta^N)^{1/2} \right) dz \\
&\leq 2\kappa_1 \int_{V_\eta^N}^\infty \exp \left(- \frac{\kappa_2 N \eta^{d \frac{1}{3}} z}{\sup_{y \in \text{Supp}(K)} \mu_T(x-y) |K|_{L^2}^2 + |K|_\infty 3^{-1/2} z^{1/2}} \right) dz \\
&\leq 2\kappa_1 \int_{V_\eta^N}^\infty \exp \left(- \frac{\kappa_2 N \eta^d z}{6 \sup_{y \in \text{Supp}(K)} \mu_T(x-y) |K|_{L^2}^2} \right) dz + 2\kappa_1 \int_{V_\eta^N}^\infty \exp \left(- \frac{\kappa_2 N \eta^d z^{1/2}}{2\sqrt{3} |K|_\infty} \right) dz \\
&\leq \frac{12\kappa_1}{\kappa_2} \sup_{y \in \text{Supp}(K)} \mu_T(x-y) |K|_{L^2}^2 (N \eta^d)^{-1} N^{-\kappa_2 \varpi / 6} \sup_{y \in \text{Supp}(K)} \mu_T(x-y) \\
&\quad + \frac{48\sqrt{3}\kappa_1}{\kappa_2} |K|_\infty |K|_{L^2} \varpi^{1/2} (N \eta^d)^{-3/2} (\log N)^{1/2} \exp \left(- \frac{\varpi^{1/2} \kappa_2 |K|_{L^2}}{2\sqrt{3} |K|_\infty} (N \eta^d)^{1/2} (\log N)^{1/2} \right),
\end{aligned}$$

where we used $\int_\nu^\infty \exp(-z^{1/2}) dz \leq 4\nu^{1/2} \exp(-\nu^{1/2})$ for $\nu \geq 16$. If

$$\varpi \geq 12\kappa_2^{-1} \max \left(\sup_{x \in \mathcal{K}, y \in \text{Supp}(K)} \mu_T(x-y), 4|K|_{L^2}^{-2} |K|_\infty^2 \kappa_2^{-1} \right)$$

and $\eta^d \geq N^{-1} \log N \frac{3 \log 2 |K|_\infty^2}{2|K|_{L^2}^2}$, we have

$$\mathbb{E} \left[\left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_\eta^N \right\}_+ \right] \leq \kappa_8 (\log N)^{1/2} N^{-2},$$

with $\kappa_8 = 2 \max \left(\frac{12\kappa_1}{\kappa_2} \sup_{x \in \mathcal{K}, y \in \text{Supp}(K)} \mu_T(x-y) |K|_{L^2}^2, \frac{48\sqrt{3}\kappa_1}{\kappa_2} |K|_\infty |K|_{L^2} \varpi^{1/2} \right)$.

In the same way, we have the rough estimate

$$\begin{aligned}
\mathbb{E} \left[\max_{\eta' \leq \eta} \left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta'}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_{\eta'}^N \right\}_+ \right] &\leq \sum_{\eta' \in \mathcal{H}} \mathbb{E} \left[\left\{ 3 \left(\widehat{\mu}_T^{N,h,\eta'}(x) - K_\eta \star \bar{\mu}_T^h(x) \right)^2 - \mathbf{V}_{\eta'}^N \right\}_+ \right] \\
&\leq \text{Card}(\mathcal{H}) \kappa_8 (\log N)^{1/2} N^{-2} \\
&\leq \kappa_8 (\log N)^{1/2} N^{-1},
\end{aligned}$$

using the previous bound. We thus have proved

$$\begin{aligned}
\mathbb{E} \left[\mathbf{A}_\eta^N \right] &\leq 2\kappa_8 (\log N)^{1/2} N^{-1} + 24 \left(\kappa_{10}^2 |K|_{L^1}^2 h^2 + \mathcal{B}_\eta(x, \mu_T)^2 \right) \\
&\leq \kappa_9 \left(\mathbf{V}_\eta^N + \mathcal{B}_\eta(x, \mu_T)^2 + h^2 \right),
\end{aligned}$$

with $\kappa_9 = \max(2\kappa_8 / |K|_{L^2}^2 \varpi, 24\kappa_{10}^2 |K|_{L^1}^2, 24)$. Back to the first step of the proof, we infer

$$\begin{aligned}
\mathbb{E} \left[\left(\widehat{\mu}_T^{N,h,\widehat{\eta}^N(T,x)}(x) - \mu_T(x) \right)^2 \right] &\leq \kappa_7 \left(\mathbb{E} \left[\mathbf{A}_\eta^N \right] + \mathbf{V}_\eta^N + \mathcal{B}_\eta(x, \mu_T)^2 + h^2 \right) \\
&\leq \kappa_5 \left(\mathbf{V}_\eta^N + \mathcal{B}_\eta(x, \mu_T)^2 + h^2 \right),
\end{aligned}$$

where now $\kappa_5 = \kappa_7(1 + \kappa_9)$. Since $\eta \in \mathcal{H}$ is arbitrary, the proof of Theorem 2.9 is complete.

Proof of Corollary 2.10. For every $\eta \in \mathcal{H}$, we have

$$\mathcal{B}_\eta(\mu_T, x)^2 \leq |\mu_T|_{k,K}^2 \eta^{2 \min(k, \ell+1)},$$

where $|\mu_T|_{k,K}$ is defined via (4.22), see (4.23) in the proof of Corollary 2.8. By Theorem 2.9, we thus obtain

$$\begin{aligned} \mathbb{E} \left[\left(\widehat{\mu}_T^{N,h, \widehat{\eta}^N(T,x)}(x) - \mu_T(x) \right)^2 \right] &\leq \kappa_5 \max(|\mu_T|_{k,K}^2, \varpi |K|_{L^2}^2, 1) \\ &\quad \times \left(\min_{\eta \in \mathcal{H}} \left((N\eta^d)^{-1} \log N + \eta^{2 \min(k, \ell+1)} \right) + h^2 \right) \\ &\leq \kappa_6 \left(\left(\frac{\log N}{N} \right)^{\frac{2 \min(k, \ell+1)}{2 \min(k, \ell+1)+d}} + h^2 \right) \end{aligned}$$

with $\kappa_6 = 2\kappa_5 \max(|\mu_T|_{k,K}^2, \varpi |K|_{L^2}^2, 1)$, using the fact that the minimum in $\eta \in \mathcal{H}$ is attained for $\eta = (N/\log N)^{1/(2 \min(k, \ell+1)+d)}$. \blacksquare

4.4. Proof of Theorem 2.11

We briefly outline the changes that are necessary in the previous proofs to extend the results to a nonlinear drift in the measure argument that satisfies Assumption A4.

Theorem 2.6 under Assumption A4

Only Lemma 4.2 needs to be extended to the case of a nonlinear drift in order to obtain Proposition 4.1, the rest of the proof remains unchanged. This amounts to prove that the random variables $b(\underline{s}, Y_{\underline{s}}^{n,h}, \widetilde{\mu}_{\underline{s}}^{N,h}) - b(\underline{s}, Y_{\underline{s}}^{n,h}, \bar{\mu}_{\underline{s}}^h)$ are sub-Gaussian, with the correct order in N . We may then repeat part of the proof of Proposition 19 of [18]. The smoothness assumption (2.14) for the nonlinear drift enables one to obtain

$$\begin{aligned} &|b(\underline{s}, Y_{\underline{s}}^{n,h}, \widetilde{\mu}_{\underline{s}}^{N,h}) - b(\underline{s}, Y_{\underline{s}}^{n,h}, \bar{\mu}_{\underline{s}}^h)|^2 \\ &\leq \sum_{l=1}^{k-1} \left| \int_{(\mathbb{R}^d)^l} \delta_\mu^l b(s, Y_{\underline{s}}^{n,h}, y^l, \bar{\mu}_{\underline{s}}^h) (\widetilde{\mu}_{\underline{s}}^{N,h} - \bar{\mu}_{\underline{s}}^h)^{\otimes l}(dy^l) \right|^2 + \min(\mathcal{W}_1(\widetilde{\mu}_{\underline{s}}^{N,h}, \bar{\mu}_{\underline{s}}^h)^2, \mathcal{W}_1(\widetilde{\mu}_{\underline{s}}^{N,h}, \bar{\mu}_{\underline{s}}^h)^{2k}), \end{aligned}$$

following the proof of Lemma 21 in [18]. Next, $\min(\mathcal{W}_1(\widetilde{\mu}_{\underline{s}}^{N,h}, \bar{\mu}_{\underline{s}}^h)^2, \mathcal{W}_1(\widetilde{\mu}_{\underline{s}}^{N,h}, \bar{\mu}_{\underline{s}}^h)^{2k})$ is sub-Gaussian with the right order in N , thanks to the sharp deviation estimates of Theorem 2 in Fournier and Guillin [20]. As for the main part of the previous expansion, we use a bound of the form

$$\mathbb{E} \left[\left| \int_{(\mathbb{R}^d)^l} \delta_\mu^l b(s, Y_{\underline{s}}^{n,h}, y^l, \bar{\mu}_{\underline{s}}^h) (\widetilde{\mu}_{\underline{s}}^{N,h} - \bar{\mu}_{\underline{s}}^h)^{\otimes l}(dy^l) \right|^{2p} \right] \leq c^p N^{-p} p! \quad \text{for every } p \geq 1,$$

for some constant c depending on b , see Lemma 22 in [18], showing eventually that this term is sub-Gaussian with the right order in N . Lemma 4.2 follows.

Theorem 2.7 under Assumption A4

Again, only an extension of Proposition 4.3 is needed, the rest of the proof remains unchanged. This only amounts to show that $(t, x) \mapsto b(t, x, \mu_t)$ is $\mathcal{C}_b^{1,3}$. The smoothness in x is straightforward, and only the proof that $t \mapsto b(t, x, \mu_t)$ is \mathcal{C}_b^1 is required. By Itô's formula, we formally have

$$\frac{d}{dt} b(t, \cdot, \mu_t) = \partial_1 b(t, \cdot, \mu_t) + \mathbb{E}[\mathcal{A}_t \delta_\mu b(t, \cdot, X_t, \mu_t)],$$

where $(X_t)_{t \in [0, T]}$ is a solution to (1.2). Assumption A4 enables one to conclude that $\frac{d}{dt}b(t, \cdot, \mu_t)$ is continuous and bounded.

Remaining proofs

The proofs of Corollary 2.8, Theorem 2.9 and Corollary 2.10 remain unchanged under Assumption A4.

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