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Some stability results for frictionless contact problems in elastodynamics formulated with Nitsche's method

Franz Chouly [1]
Hao Huang [2]
Nicolas Pignet [3]

[1] Centro de Matemática (CMAT), IRL-2030 CNRS IFUMI, Facultad de Ciencias, Universidad de la República, 11400 Montevideo, Uruguay.
E-mail address: fchouly@cmat.edu.uy
[2] EDF Lab Paris-Saclay, 7 boulevard Gaspard Monge, 91120 Palaiseau, France — Institut de Mathématiques de Bourgogne, Université de Bourgogne, 21078 Dijon, France.
[3] EDF Lab Paris-Saclay, 7 boulevard Gaspard Monge, 91120 Palaiseau, France.

Abstract. This work focuses on the numerical performance of the Nitsche-based Finite Element Method for dynamic unilateral contact problems combined with two implicit one-step time-marching schemes. The non-linear contact boundary conditions cause irregularities, which may lead to unstable performance and potential divergence during simulations. By focusing on the discontinuities inherent in dynamic contact problems, we provide new stability results for the proposed methods.

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Keywords. Time-marching schemes, Nitsche's method, unilateral contact, finite element method, elastodynamics.

1. Introduction

Contact and friction in elastodynamics are usually solved using the method of lines, combining the Finite Element Method (FEM) in space and a Finite Difference Method (FDM) as a time-marching scheme [53, 57]. This paper is focused on this aforementioned approach and investigates more in depth some aspects for a particular class of FEM-FDM approximation methods. Notably, for frictionless elastodynamic contact, practical challenges are to avoid spurious oscillations on the displacement, velocity and stress, and to preserve the mechanical energy of the system [38]. For the sake of exhaustibility, let us mention a few alternatives, like space-time discretization techniques [4, 43, 55], techniques from non-smooth rigid bodies dynamics [2, 3, 60] or smoothed particle hydrodynamics [21, 59]. For the space semi-discretization of contact, the FEM can be substituted with other discrete variational methods such as Boundary Element methods [8, 9], Isogeometric methods [36, 50], Discontinuous Galerkin methods [44, 64, 65], Virtual Element methods [6, 34, 39, 47, 56, 70, 71, 72, 73] or Hybrid High Order (HHO) methods [22, 24].

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Originally devised for the Dirichlet boundary conditions [23, 62, 67], Nitsche's method has been introduced in [26] for Signorini's problem in elastostatics, and later on extended to elastodynamics [29, 30]. It has been combined with various time-marching schemes, first the families of θ-scheme, of Newmark schemes and a new hybrid scheme in [30]. Later on it has been combined with the explicit Verlet scheme [33], implicit-explicit IMEX schemes (the implicit term being linear) [16] and to the Hilber-Hughes-Taylor HHT-α scheme and Bathe TR-BDF2 scheme [51].

Signorini contact in elastodynamics, when discretized in space with a standard finite element method, leads to an ill-posed differential inclusion with multiple solutions [52]. If this differential inclusion is combined with a standard time-marching scheme like Crank–Nicolson, this ill-posedness leads to poor numerical results, with spurious oscillations and no energy conservation. Notably, this effect is more important when the time-step is reduced. This ill-posedness at the semi-discrete level can be fixed using a specific treatment of contact, such as the modified mass method, a penalty/regularization or Nitsche's method [29, 38, 52]. Moreover, these methods preserve a semi-discrete counterpart of the mechanical energy. However, even in this situation, standard time-marching schemes such as Crank–Nicolson do not preserve a fully discrete energy, due to the Signorini contact conditions, and some special numerical schemes can be designed to remedy to this issue, see for instance [48] for penalty or [30] for Nitsche.

In this work, we study the mathematical properties of two families of time-marching schemes combined with Nitsche-based finite element method for frictionless unilateral contact problems. Namely, we consider theta-schemes and Newmark schemes. The enforcement of contact conditions is carried out with the symmetric Nitsche method. Initially introduced by J.A. Nitsche in [62] (see also [23, 67]), the Nitsche method has been extended in [26, 31] for the Signorini problem (unilateral contact problem in elastostatics), and then [28, 29, 30] for elastodynamics. For the derivation of Nitsche formulation in the context of inequalities and the link with other methods, see, e.g., [13, 18, 20, 25, 31, 32, 50]. Nitsche's method is closely related to stabilized mixed methods, specifically to the Barbosa–Hugues stabilization [10, 11, 12, 49]. However it does not introduce a Lagrange multiplier as an extra unknown conversely to mixed/mortar methods [1, 14, 69] or Augmented Lagrangian technologies [5, 19, 66]. It differs also from penalty techniques [27, 37, 40, 45, 53, 54, 63] because it preserves consistency and is more robust with respect to the numerical parameters.

Particularly we revisit stability results provided in [29, 30]. We take advantage of a new semi-discrete energy proposed by E. Burman, M.A. Fernandez and Stefan Frei in [17], that we will call BFF energy for the sake of conciseness. This discrete energy has the remarkable property of being always nonnegative whatever the values of the discretization parameters are, notably the Nitsche parameter that does not need to be large. In [17] a stability bound has been proven at the semi-discrete level. We propose an adaptation of this energy at the fully discrete level and provide new bounds for the evolution of energy for the aforementioned fully discrete methods. We complement this stability study with some new numerical experiments.

We first introduce the definition of several discrete energies in Section 2, then we define the time-marching schemes used in Section 3. In Section 4, we present a stability analysis of fully discrete schemes considered previously. Finally, we provide some numerical illustrations in Section 5.

2. Nitsche-based finite element method for elasto-dynamic unilateral contact problems

We consider an elastic body $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, see Figure 2.1. On the boundary $\partial\Omega := \Gamma_D \cup \Gamma_N \cup \Gamma_C$ of Ω, Dirichlet, Neumann, and Signorini boundary conditions are applied respectively on the disjoint subsets Γ_D, Γ_N, and Γ_C.

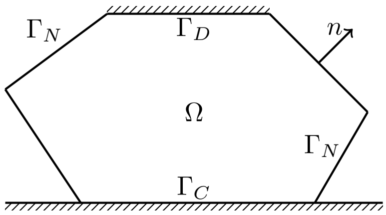


FIGURE 2.1. Elastic body occupying the domain Ω, with the boundary $\partial\Omega$.

We seek the displacement field $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^d$, where $T > 0$ is the final time, verifying the equations (2.1):

$$\begin{aligned}
&\rho\ddot{\mathbf{u}} - \operatorname{div}(\sigma(\mathbf{u})) = \mathbf{f}, && \text{in } \Omega \times (0,T), && \text{(i)}\\
&\sigma(\mathbf{u}) = \lambda \operatorname{tr}(\epsilon(\mathbf{u}))\mathbf{I} + 2\mu\epsilon(\mathbf{u}), && \text{in } \Omega \times (0,T), && \text{(ii)}\\
&\mathbf{u} = \mathbf{0}, && \text{on } \Gamma_D \times (0,T), && \text{(iii)}\\
&\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{f}_N, && \text{on } \Gamma_N \times (0,T), && \text{(iv)}\\
&u_n \le 0, \quad \sigma_n(\mathbf{u}) \le 0, \quad u_n\sigma_n(\mathbf{u}) = 0, && \text{on } \Gamma_C \times (0,T), && \text{(v)}\\
&\sigma(\mathbf{u}) \cdot \mathbf{n} - \sigma_n(\mathbf{u})\mathbf{n} = \mathbf{0}, && \text{on } \Gamma_C \times (0,T), && \text{(vi)}\\
&\mathbf{u}(\cdot,0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(\cdot,0) = \dot{\mathbf{u}}_0, && \text{in } \Omega. && \text{(vii)}
\end{aligned} \tag{2.1}$$

The equations (i)–(iv) describe the problem of elastodynamics in small deformation, where ρ is the mass density, $\mathbf{f}$ is the volumetric source term, $\mathbf{f}_N$ is the surface charge, λ and μ are Lamé coefficients, $\operatorname{tr}(\,\cdot\,)$ denotes the trace of a matrix, $\epsilon(\,\cdot\,) := \frac{1}{2}\Big(\nabla(\,\cdot\,) + \nabla^T(\,\cdot\,)\Big)$ is the small deformation tensor and $\mathbf{I}$ is the identity matrix of dimension d. The equations (v) and (vi) denote Signorini's condition for frictionless contact, where $u_n = \mathbf{u} \cdot \mathbf{n}$, $\sigma_n(\mathbf{u}) = (\sigma(\mathbf{u}) \cdot \mathbf{n}) \cdot \mathbf{n}$ are respectively the normal displacement and the contact pressure at boundary Γ_C. The initial conditions at the time $t = 0$ on the initial displacement and velocity fields are given in equation (vii). We also introduce the bilinear and linear forms

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &:= (\sigma(\mathbf{u}), \epsilon(\mathbf{v}))_\Omega,\\
l(t)(\mathbf{v}) &:= (\mathbf{f}(t), \mathbf{v})_\Omega + (\mathbf{f}_N(t), \mathbf{v})_{\Gamma_N},
\end{aligned} \tag{2.2}$$

where the notation $(\mathbf{u},\mathbf{v})_\Omega := \int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\Omega$ and $(\mathbf{u},\mathbf{v})_{\Gamma_F} := \int_{\Gamma_F} \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\Gamma$ are the L^2-products defined on Ω and Γ_F respectively, with $\Gamma_F \in \{\partial\Omega, \Gamma_D, \Gamma_N, \Gamma_C\}$. The associated norms are denoted by $\|\cdot\|_\Omega := (\cdot,\cdot)_\Omega^{\frac{1}{2}}$ and $\|\cdot\|_{\Gamma_F} := (\,\cdot\,,\cdot\,)_{\Gamma_F}^{\frac{1}{2}}$.

The total mechanical energy associated with the solution $\mathbf{u}$ of dynamic Signorini Problem (2.1) is:

$$E(t) = \frac{1}{2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}\right\|_\Omega^2 + \frac{1}{2}a(\mathbf{u},\mathbf{u}), \quad \forall\, t \in [0,T]. \tag{2.3}$$

Moreover, with the persistency condition [7, 48, 58] $(\sigma(\mathbf{u}) \cdot \mathbf{n}) \cdot \dot{\mathbf{u}} = 0$ on Γ_C, and when $l(t)$ vanishes, the energy is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0.$$

2.1. Space discretization and enforcement of contact conditions

Let $\mathbf{V}^h$ be the Lagrange finite element space to discretize the displacement, of degree one or two ($k = 1$ or 2), and based on a mesh $\mathcal{T}^h$ of the domain Ω:

$$\mathbf{V}^h := \left\{ \mathbf{v}^h \in \left(\mathscr{C}^0(\overline{\Omega})\right)^d : \mathbf{v}^h_{|\Gamma_D} = \mathbf{0};\ \mathbf{v}^h_{|T} = \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}^h \right\}.$$

The enforcement of contact conditions is introduced at the semi-discrete level (semi-discretization in space). We focus on the symmetric Nitsche approximation of contact and follow [32].

We define the space-discrete weak formulation of the Nitsche-FEM method. First, we introduce the linear discrete operator

$$P_{\Theta_N,\gamma_N} : \begin{array}{l} \mathbf{V}^h \longrightarrow L^2(\Gamma_C) \\ \mathbf{v}^h \longmapsto \Theta_N \sigma_n(\mathbf{v}^h) - \gamma_N v^h_n \end{array}, \tag{2.4}$$

where $\Theta_N \in \{-1, 0, 1\}$ is a fixed parameter and γ_N is a positive function independent of $\mathbf{v}$, such that

$$\gamma_{N|T\cap\Gamma_C} = \frac{\gamma_0}{h_T},$$

for each triangle T intersected with Γ_C, where h_T is the diameter of the triangle T and γ_0 is a positive constant (the Nitsche parameter). The primal discrete problem is as follows:

$$\begin{cases} \text{Seek}\, \mathbf{u}^h : [0,T] \longrightarrow \mathbf{V}^h, \ \text{s.t.} \\ \left(\rho \ddot{\mathbf{u}}^h(t), \mathbf{v}^h\right)_\Omega + a_{\Theta_N,\gamma_N}(\mathbf{u}^h(t), \mathbf{v}^h) + \left(\dfrac{1}{\gamma_N}\left[P_{1,\gamma_N}(\mathbf{u}^h(t))\right]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right)_{\Gamma_C} = l(t)(\mathbf{v}), \end{cases} \tag{2.5}$$

for every $\mathbf{v}^h \in \mathbf{V}^h$ and where $a_{\Theta_N,\gamma_N}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \left(\frac{\Theta_N}{\gamma_N}\sigma_n(u^h), \sigma_n(v^h)\right)_{\Gamma_C}$. The notations $[\,\cdot\,]_{\mathbb{R}^-} := \min(0, \cdot\,)$ and $[\,\cdot\,]_{\mathbb{R}^+} := \max(0, \cdot\,)$ denote the projection on the half-line formed by negative or positive real numbers. The static counterpart of this method is well-posed and optimally convergent when the mesh size h vanishes, provided that the Nitsche parameter γ_0 is large enough. Subsequently, by using Riesz's representation theorem, the mass operator $\mathbf{M} : \mathbf{V}^h \to \mathbf{V}^h$ and the nonlinear operator $\mathbf{B}_N : \mathbf{V}^h \to \mathbf{V}^h$ are defined such that

$$\begin{aligned} \left(\mathbf{M}(\mathbf{v}^h), \mathbf{w}^h\right)_\Omega &= \left(\rho \mathbf{v}^h, \mathbf{w}^h\right)_\Omega, \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h, \\ \left(\mathbf{B}_N(\mathbf{v}^h), \mathbf{w}^h\right)_\Omega &= a_{\Theta_N,\gamma_N}(\mathbf{v}^h, \mathbf{w}^h) + \left(\frac{1}{\gamma_N}\left[P_{1,\gamma_N}(\mathbf{v}^h)\right]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\mathbf{w}^h)\right)_{\Gamma_C}, \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h. \end{aligned}$$

The vector $\mathbf{L}(t) \in \mathbf{V}^h$ is then defined such that

$$\left(\mathbf{L}(t), \mathbf{w}^h\right)_\Omega = l(t)(\mathbf{w}^h), \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h.$$

Then, the compact form of (2.5) is

$$\begin{cases} \text{Seek}\, \mathbf{u}^h : [0,T] \longrightarrow \mathbf{V}^h, \ \text{s.t.} \\ \mathbf{M}(\ddot{\mathbf{u}}^h(t)) + \mathbf{B}_N(\mathbf{u}^h(t)) = \mathbf{L}(t), \\ \mathbf{u}^h(0) = \mathbf{u}^h_0, \quad \dot{\mathbf{u}}^h(0) = \dot{\mathbf{u}}^h_0, \end{cases}$$

which is a system of Lipschitz differential equations and its well-posedness has been proven in [29]. Note that, though the standard convergence analysis in the static case requires, for technical reasons, extra regularity assumptions [31], there is no extra regularity required to write a Nitsche method (and the assumptions in [31] can be weakened in fact, see [42, 46]).

2.2. Preliminary results

The following two properties are classical for the negative part:

$$[a]_{\mathbb{R}^-} \le a, \quad a[a]_{\mathbb{R}^-} = [a]_{\mathbb{R}^-}^2, \quad \forall\, a \in \mathbb{R},$$
$$([a]_{\mathbb{R}^-} - [b]_{\mathbb{R}^-})(a-b) \ge ([a]_{\mathbb{R}^-} - [b]_{\mathbb{R}^-})^2 \ge 0, \quad \forall\, a, b \in \mathbb{R}.$$

The lemma below is helpful as an intermediate result.

Lemma 2.1. *For $a, b \in \mathbb{R}$, the following inequality holds:*

$$0 \le -[a]_{\mathbb{R}^+}[b]_{\mathbb{R}^-} \le \frac{1}{4}(a-b)^2. \tag{2.6}$$

Proof. When $a \le 0$ or $b \ge 0$, we have $-[a]_{\mathbb{R}^+}[b]_{\mathbb{R}^-} = 0 \le 1/4(a-b)^2$. For the case $a > 0$ and $b < 0$, it leads to $-[a]_{\mathbb{R}^+}[b]_{\mathbb{R}^-} = |a| \cdot |b| > 0$. With the relation

$$(|a| + |b|)^2 - 4|a| \cdot |b| = (|a| - |b|)^2 \ge 0,$$

we can conclude that for this case:

$$-[a]_{\mathbb{R}^+}[b]_{\mathbb{R}^-} = |a| \cdot |b| \le \frac{1}{4}(|a| + |b|)^2 = \frac{1}{4}(a-b)^2.$$

■

Another useful property is the following:

Lemma 2.2. *For $a : t \mapsto a(t)$ a differentiable function the following equality holds:*

$$[a]_{\mathbb{R}^-} \frac{\mathrm{d}[a]_{\mathbb{R}^-}}{\mathrm{d}t} = [a]_{\mathbb{R}^-} \frac{\mathrm{d}a}{\mathrm{d}t}. \tag{2.7}$$

Proof. First if $a(t) = 0$, then the equality 2.7 holds. Suppose that $a(t) \ne 0$. We introduce $H(\,\cdot\,)$ the Heaviside function: for $x \in \mathbb{R}$,

$$H(x) = 0 \text{ if } x < 0, \qquad H(x) = 1 \text{ if } x \ge 0.$$

Since $[a(t)]_{\mathbb{R}^-} = H(-a(t))a(t)$, we use the chain rule and there holds

$$[a(t)]_{\mathbb{R}^-} \frac{\mathrm{d}[a]_{\mathbb{R}^-}}{\mathrm{d}t}(t) = [a(t)]_{\mathbb{R}^-} \frac{\mathrm{d}}{\mathrm{d}t}(H(-a(t))a(t)) = [a(t)]_{\mathbb{R}^-} \Big(H(-a(t)) \frac{\mathrm{d}a}{\mathrm{d}t}(t) + a(t) \frac{\mathrm{d}H(-a)}{\mathrm{d}t}(t) \Big).$$

We finally use the properties $\mathrm{d}H(-a)/\mathrm{d}t = 0$ and $[a]_{\mathbb{R}^-} H(-a) = [a]_{\mathbb{R}^-}$:

$$[a(t)]_{\mathbb{R}^-} \frac{\mathrm{d}[a]_{\mathbb{R}^-}}{\mathrm{d}t}(t) = [a(t)]_{\mathbb{R}^-} H(-a(t)) \frac{\mathrm{d}a}{\mathrm{d}t}(t) = [a(t)]_{\mathbb{R}^-} \frac{\mathrm{d}a}{\mathrm{d}t}(t).$$

This ends the proof. ■

We need the following inequalities for the space discretization. The first discrete trace inequality has been proven in [31] (see alternatively [68, Lemma 2.1] for the scalar case).

Lemma 2.3 (First discrete trace inequality). *There exists $C_{tr} > 0$, independent of the parameter γ_0 and of the mesh size h, such that:*

$$\left\| \gamma_N^{-\frac{1}{2}} \sigma_n(\mathbf{v}^h) \right\|_{\Gamma_C}^2 \le C_{tr} \gamma_0^{-1} \left\| \mathbf{v}^h \right\|_{H^1(\Omega)}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \tag{2.8}$$

The inverse inequality below is proven for instance in [41, Corollary 1.141, Remark 1.143, p. 76–77].

Lemma 2.4 (Inverse inequality). *Suppose that the mesh $\mathcal{T}^h$ is quasi-uniform, then there exists $C_{inv} > 0$ independent of the mesh size h, such that:*

$$\left\| \mathbf{v}^h \right\|_{H^1(\Omega)} \le C_{inv} h^{-1} \left\| \mathbf{v}^h \right\|_{\Omega}, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \tag{2.9}$$

We finally prove a second discrete trace inequality.

Lemma 2.5 (Second discrete trace inequality). *Suppose that the mesh $\mathcal{T}^h$ is quasi-uniform, then there exists $C > 0$, independent of γ_0 and h, such that:*

$$\left\|\gamma_N^{\frac{1}{2}} v_n^h\right\|_{\Gamma_C}^2 \leq C\frac{\gamma_0}{h^2}\left\|\mathbf{v}^h\right\|_{\Omega}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \tag{2.10}$$

and that:

$$\left\|\gamma_N^{\frac{1}{2}} v_n^h\right\|_{\Gamma_C}^2 \leq C\frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \tag{2.11}$$

Proof. We use [15, Theorem 1.6.6] and inverse inequality (2.9) to have the first inequality:

$$\left\|\gamma_N^{\frac{1}{2}} v_n^h\right\|_{\Gamma_C}^2 \leq \frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{\partial\Omega}^2 \leq C\frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}\left\|\mathbf{v}^h\right\|_{\Omega} \leq C\frac{\gamma_0}{h^2}\left\|\mathbf{v}^h\right\|_{\Omega}^2.$$

Moreover, we use the relation $\|\cdot\|_{\Omega} \leq \|\cdot\|_{H^1(\Omega)}$ to prove the second inequality

$$C\frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}\left\|\mathbf{v}^h\right\|_{\Omega} \leq C\frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2.$$ ■

We can also bound the operator P_{Θ_N,γ_N} as follows.

Lemma 2.6 (Bounds on the operator P_{Θ_N,γ_N}). *Suppose that the mesh $\mathcal{T}^h$ is quasi-uniform, and that γ_0 is large enough, then there exists $C > 0$, independent of γ_0 and h, such that:*

$$\left\|\gamma_N^{-\frac{1}{2}} P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right\|_{\Gamma_C}^2 \leq C\frac{\gamma_0}{h^2}\left\|\mathbf{v}^h\right\|_{\Omega}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h, \tag{2.12}$$

and that:

$$\left\|\gamma_N^{-\frac{1}{2}} P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right\|_{\Gamma_C}^2 \leq C\frac{\gamma_0}{h}\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \tag{2.13}$$

Proof. We start with the triangular inequality:

$$\left\|\gamma_N^{-\frac{1}{2}} P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right\|_{\Gamma_C}^2 \leq 2\left\|\gamma_N^{-\frac{1}{2}} \sigma_n(\mathbf{v}^h)\right\|_{\Gamma_C}^2 + 2\left\|\gamma_N^{\frac{1}{2}} v_n^h\right\|_{\Gamma_C}^2,$$

then we use the first and second discrete trace inequalities (2.3), (2.5), and the fact that γ_0 is large enough, to conclude the proof. ■

We finally have the coercivity of the bilinear form a_{Θ_N,γ_N}:

Lemma 2.7 (Coercivity of the bilinear form a_{Θ_N,γ_N}). *Suppose that the mesh $\mathcal{T}^h$ is quasi-uniform then there exists $C > 0$, independent of h, such that:*

$$a_{\Theta_N,\gamma_N}(\mathbf{v}^h, \mathbf{v}^h) \geq \left(C_a - \Theta_N C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Moreover, for γ_0 large enough if $\Theta_N = 1$ or $\gamma_0 > 0$ if $\Theta_N = 0$ or $\Theta_N = -1$, then there exists $C > 0$, independent of h (and also independent of γ_0 for $\Theta_N = 0$ or $\Theta_N = -1$).

$$a_{\Theta_N,\gamma_N}(\mathbf{v}^h, \mathbf{v}^h) \geq C\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2, \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h.$$

Proof. Since the operator $a(\cdot,\cdot)$ is coercive (i.e., $\exists\, C_a \geq 0$, s.t. $a(\mathbf{v}^h, \mathbf{v}^h) > C_a\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2$, for all $\mathbf{v}^h \in \mathbf{V}^h$), using the definition of a_{Θ_N,γ_N} and the first discrete trace inequality (2.8), we have

$$a_{\Theta_N,\gamma_N}(\mathbf{v}^h, \mathbf{v}^h) = a(\mathbf{v}^h, \mathbf{v}^h) - \Theta_N\left\|\gamma_N^{-\frac{1}{2}} \sigma_n(\mathbf{v}^h)\right\|_{\Gamma_C}^2 \geq \left(C_a - \Theta_N C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{v}^h\right\|_{H^1(\Omega)}^2, \tag{2.14}$$

Hence, for γ_0 large enough, i.e., $\gamma_0 > \Theta_N C_{tr} C_a^{-1}$, we have $C_a - \Theta_N C_{tr}\gamma_0^{-1} > 0$. ■

2.3. Discrete energy notations

Let us define the discrete energy as follows:

$$E^h(t) := \frac{1}{2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^h(t)\right\|_{\Omega}^2 + \frac{1}{2}a(\mathbf{u}^h(t),\mathbf{u}^h(t)),$$

which is associated to the solution $\mathbf{u}^h(t)$ of Problem (2.5). Note that this is the direct transposition of the mechanical energy $E(t)$ of the continuous system. Set also

$$\begin{aligned} E^h_{\Theta_N}(t) &:= E^h(t) - \frac{\Theta_N}{2}\left\|\gamma_N{}^{-\frac{1}{2}}\sigma_n(\mathbf{u}^h(t))\right\|_{\Gamma_C}^2 + \frac{\Theta_N}{2}\left\|\gamma_N{}^{-\frac{1}{2}}\left[P_{1,\gamma_N}(\mathbf{u}^h(t))\right]_{\mathbb{R}^-}\right\|_{\Gamma_C}^2 \\ &:= E^h(t) - \Theta_N R^h(t), \end{aligned}$$

which corresponds to a modified energy (also called augmented energy) in which a consistent term is added. This term denoted $R^h_N(t)$ represents, roughly speaking, the nonfulfillment of the contact condition (2.1) (v) by $\mathbf{u}^h(t)$ [25, 29]. When $l(t) = 0$ and $\Theta_N = 1$, we can prove that this modified energy is conserved: $E^h_{\Theta_N}(t) = E^h_{\Theta_N}(0)$, see, e.g., [29, Theorem 2.10, Corollary 2.11].

Remark 2.8 (Modified energy as an energy)**.** The aforementioned energy $E^h_{\Theta_N}(t)$ is positive for γ_0 large enough, by direct application of the discrete trace inequality (2.8). For an arbitrary value of γ_0, there is no guarantee this energy remains positive. See [33, Proposition 8] for more details.

The BFF (Burman–Fernández–Frei) energy is a non-negative energy, irrespectively of the value of the Nitsche parameter γ_0 and even for small values of this parameter. It has been recently introduced by E. Burman, M. Fernández and S. Frei in [17] for Nitsche-based fluid-structure-contact interaction problems. The BFF energy is defined as follows:

$$E^h_{BFF}(t) := E^h(t) + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\right\|_{\Gamma_C}^2.$$

We observe that this energy is also augmented with a consistent term. We expect that these supplementary terms vanish under mesh refinement. This can be proven in the static case when the solution is regular enough (in $H^s(\Omega;\mathbb{R}^d)$, $s > 3/2$). In the dynamic case, and from the theoretical viewpoint, we do not know if this property still holds and under which assumption on the solution. However, in practice, we can at least observe these supplementary terms are small (see the numerical section below).

Proposition 2.9. *Let* $\mathbf{u}$ *be a solution of Problem 2.5 with* $l(t) = 0$, $\Theta_N = 1$, *and assume that* $\gamma_0 \geq 4C_{tr}C_a^{-1}$, *then the BFF energy remains bounded for* $t \in (0,T)$*:*

$$\frac{1}{2}E^h_{BFF}(t) \leq E^h_{BFF}(0) + \left(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(0)), ([P_{1,\gamma_N}(\mathbf{u}^h(0))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(0)))\right)_{\Gamma_C}.$$

Proof. We adapt the proof of [17] to our problem where there is no fluid. Firstly using the definition of P_{Θ_N,γ_N} multiple times and $\partial/\partial t\|a\|^2_{\Gamma_C} = 2(a,\dot{a})_{\Gamma_C}$, we have,

$$\begin{aligned} &\left(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\dot{\mathbf{u}}^h)\right)_{\Gamma_C} - \left(\frac{\Theta_N}{\gamma_N}\sigma_n(\mathbf{u}^h),\sigma_n(\dot{\mathbf{u}}^h)\right)_{\Gamma_C} \\ &= \left(\frac{\Theta_N}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)),\sigma_n(\dot{\mathbf{u}}^h)\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-},\gamma_N\dot{u}^h\right)_{\Gamma_C} \\ &= \left(\frac{\Theta_N}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)),\sigma_n(\dot{\mathbf{u}}^h)\right)_{\Gamma_C} + \left(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, P_{1,\gamma_N}(\dot{\mathbf{u}}^h) - \sigma_n(\dot{\mathbf{u}}^h)\right)_{\Gamma_C} \end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{\Theta_N}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} + \Big(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, \partial_t(P_{1,\gamma_N}(\mathbf{u}^h) - \sigma_n(\mathbf{u}^h))\Big)_{\Gamma_C} \\
&= \Big(\frac{\Theta_N}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} + \Big(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, \partial_t([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big)_{\Gamma_C} \\
&= \Big(\frac{\Theta_N}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} + \frac{1}{2}\frac{\partial}{\partial t}\Big\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big\|_{\Gamma_C}^2 \\
&\quad + \Big(\sigma_n(\mathbf{u}^h), \partial_t([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big)_{\Gamma_C} \\
&= \frac{1}{2}\frac{\partial}{\partial t}\Big\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big\|_{\Gamma_C}^2 + \frac{\partial}{\partial t}\Big(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h), [P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)\Big)_{\Gamma_C} \\
&\quad + \Big(\frac{\Theta_N - 1}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C}.
\end{aligned}
$$

where Lemma 2.2 has been used in the fifth line.
Secondly, testing (2.5) with $\mathbf{v}^h = \dot{\mathbf{u}}^h$, integrating in time on $(0,t) \subset (0,T]$ and using previous equality, we get

$$
\begin{aligned}
0 &= \int_0^t \Big(\rho\ddot{\mathbf{u}}^h, \dot{\mathbf{u}}^h\Big)_{\Omega} \mathrm{d}t + \int_0^t a(\mathbf{u}^h, \dot{\mathbf{u}}^h)\mathrm{d}t - \int_0^t \Big(\frac{\Theta_N}{\gamma_N}\sigma_n(\mathbf{u}^h), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} \mathrm{d}t \\
&\quad + \int_0^t \Big(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} \mathrm{d}t \\
&= \frac{1}{2}\Big[\Big\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^h\Big\|_{\Omega}^2\Big]_0^t + \frac{1}{2}\Big[a(\mathbf{u}^h, \mathbf{u}^h)\Big]_0^t - \int_0^t \Big(\frac{\Theta_N}{\gamma_N}\sigma_n(\mathbf{u}^h), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} \mathrm{d}t \\
&\quad + \int_0^t \Big(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} \mathrm{d}t \\
&= \frac{1}{2}\Big[\Big\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^h\Big\|_{\Omega}^2\Big]_0^t + \frac{1}{2}\Big[a(\mathbf{u}^h, \mathbf{u}^h)\Big]_0^t + \frac{1}{2}\Big[\Big\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big\|_{\Gamma_C}^2\Big]_0^t \\
&\quad + \Big[\Big(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h), ([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h))\Big)_{\Gamma_C}\Big]_0^t \\
&\quad + \int_0^t \Big(\frac{\Theta_N - 1}{\gamma_N}([P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h)), \sigma_n(\dot{\mathbf{u}}^h)\Big)_{\Gamma_C} \mathrm{d}t.
\end{aligned}
$$

For the symmetric version ($\Theta_N = 1$), the last term vanishes and we have:

$$
\begin{aligned}
E_{BFF}^h(t) = E_{BFF}^h(0) &+ \Big(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(0)), ([P_{1,\gamma_N}(\mathbf{u}^h(0))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(0)))\Big)_{\Gamma_C} \\
&- \Big(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(t)), ([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\Big)_{\Gamma_C}.
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality and the Young's inequality, we have:

$$
\begin{aligned}
&\left|\Big(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(t)), ([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\Big)_{\Gamma_C}\right| \\
&\qquad \le \Big\|\gamma_N^{-\frac{1}{2}}\sigma_n(\mathbf{u}^h(t))\Big\|_{\Gamma_C}^2 + \frac{1}{4}\Big\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\Big\|_{\Gamma_C}^2.
\end{aligned}
$$

Using the first discrete trace inequality (2.8), we have:

$$E^h_{BFF}(t) \leq E^h_{BFF}(0) + \left(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(0)), ([P_{1,\gamma_N}(\mathbf{u}^h(0))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(0)))\right)_{\Gamma_C} + C_{tr}\gamma_0^{-1}\left\|\mathbf{u}^h(t)\right\|^2_{H^1(\Omega)} + \frac{1}{4}\left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\right\|^2_{\Gamma_C}. \quad (2.15)$$

From the definition of the BFF energy, the coercivity of the bilinear form $a(\cdot,\cdot)$ and under the assumption that $\gamma_0 \geq 4C_{tr}C_a^{-1}$, the following holds:

$$\begin{aligned}
2E^h_{BFF}(t) &= a(\mathbf{u}^h(t), \mathbf{u}^h(t)) + \left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^h(t)\right\|^2_{\Omega} + \left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\right\|^2_{\Gamma_C} \\
&\geq a(\mathbf{u}^h(t), \mathbf{u}^h(t)) + \left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\right\|^2_{\Gamma_C} \\
&\geq C_a\left\|\mathbf{u}^h(t)\right\|^2_{H^1(\Omega)} + \left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\right\|^2_{\Gamma_C} \\
&\geq 4C_{tr}\gamma_0^{-1}\left\|\mathbf{u}^h(t)\right\|^2_{H^1(\Omega)} + \left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^h(t))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(t)))\right\|^2_{\Gamma_C}.
\end{aligned} \quad (2.16)$$

Combining (2.15) and (2.16), we have

$$\frac{1}{2}E^h_{BFF}(t) \leq E^h_{BFF}(0) + \left(\frac{1}{\gamma_N}\sigma_n(\mathbf{u}^h(0)), ([P_{1,\gamma_N}(\mathbf{u}^h(0))]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^h(0)))\right)_{\Gamma_C}.$$

The right hand-side terms depends only on the initial condition and thus, the BFF energy $E^h_{BFF}(t)$ remains bounded. ■

3. Time discretization

In this work, we focus on two time-marching schemes: the θ-scheme and the Newmark scheme. Let $\Delta t > 0$ be the time-step and consider a uniform discretization of the time interval $[0,T] : (t^0, \ldots, t^N)$, with $t^n = n\Delta t, n = 0, \ldots, N$. The schemes consist of solving a nonlinear problem for each time instant t^{n+1} using the displacement $\mathbf{u}^{h,n}$, velocity $\dot{\mathbf{u}}^{h,n}$, and acceleration $\ddot{\mathbf{u}}^{h,n}$ of instant t^n as known variables, where $\mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}$, and $\ddot{\mathbf{u}}^{h,n+1}$ are the displacement, velocity, and acceleration to be solved, and $\mathbf{L}^{n+1} = \mathbf{L}(t^{n+1})$. The initial conditions are $\mathbf{u}^{h,0} = \mathbf{u}^h_0, \dot{\mathbf{u}}^{h,0} = \dot{\mathbf{u}}^h_0, \ddot{\mathbf{u}}^{h,0} = \ddot{\mathbf{u}}^h_0$. We note that the initial acceleration is calculated by the relation $\mathbf{M}\ddot{\mathbf{u}}^{h,0} + \mathbf{B}_N(\mathbf{u}^{h,0}) = \mathbf{L}^0$.

3.1. θ-scheme

The fully discrete formulation for Problem (2.5) is such that:

$$\begin{cases}
\text{Seek } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ s.t.} \\
\mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \Delta t((1-\theta)\dot{\mathbf{u}}^{h,n} + \theta\dot{\mathbf{u}}^{h,n+1}), & \text{(i)} \\
\dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \Delta t((1-\theta)\ddot{\mathbf{u}}^{h,n} + \theta\ddot{\mathbf{u}}^{h,n+1}), & \text{(ii)} \\
\mathbf{M}\ddot{\mathbf{u}}^{h,n+1} + \mathbf{B}_N(\mathbf{u}^{h,n+1}) = \mathbf{L}^{n+1}. & \text{(iii)}
\end{cases} \quad (3.1)$$

with the parameter $\theta \in [0,1]$. The two classical schemes are forward, respectively backward, Euler schemes for value $\theta = 0$, respectively $\theta = 1$. We recall that Θ_N is the parameter for the Nitsche's variant.

3.2. Newmark scheme

The Newmark scheme [61] is another classical scheme for elastodynamic problems. Its fully discrete formulation reads:

$$\begin{cases} \text{Seek } \mathbf{u}^{h,n+1}, \dot{\mathbf{u}}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} \in \mathbf{V}^h \text{ s.t.} \\ \mathbf{u}^{h,n+1} = \mathbf{u}^{h,n} + \Delta t \dot{\mathbf{u}}^{h,n} + \dfrac{\Delta t^2}{2}((1-2\beta)\ddot{\mathbf{u}}^{h,n} + 2\beta \ddot{\mathbf{u}}^{h,n+1}), & \text{(i)} \\ \dot{\mathbf{u}}^{h,n+1} = \dot{\mathbf{u}}^{h,n} + \Delta t((1-\zeta)\ddot{\mathbf{u}}^{h,n} + \zeta \ddot{\mathbf{u}}^{h,n+1}). & \text{(ii)} \\ \mathbf{M}\ddot{\mathbf{u}}^{h,n+1} + \mathbf{B}_N(\mathbf{u}^{h,n+1}) = \mathbf{L}^{n+1}. & \text{(iii)} \end{cases} \tag{3.2}$$

The parameters (β, ζ) determine the numerical performance of the method, including the stability, accuracy, and damping properties. $\beta = 0$, $\zeta = \frac{1}{2}$ yields an explicit central difference scheme; Crank–Nicolson scheme is obtained by taking $\beta = \frac{1}{4}$, $\zeta = \frac{1}{2}$ (which is equivalent to θ-scheme with $\theta = \frac{1}{2}$). The Crank–Nicolson scheme is unconditionally stable and has a second-order accuracy for linear elastodynamic (without contact conditions). However, in the presence of contact, it can be more complicated and can introduce additional oscillations and numerical energy [30]. The Newmark scheme could be seen as the case of $\alpha = 0$ for the HHT-α scheme (Hilber-Hughes-Taylor-α scheme).

The well-posedness of Problem (3.1) (θ-scheme), respectively Problem (3.2) (Newmark scheme), has been proven in [29] under the condition $(1+\Theta_N)^2\gamma_0^{-1} \le C_{wp}\Big(1 + \frac{\rho h^2}{\theta^2 \Delta t^2}\Big)$ with C_{wp} a positive constant, respectively $(1+\Theta_N)^2\gamma_0^{-1} \le C_{wp}\Big(1 + \frac{\rho h^2}{\beta \Delta t^2}\Big)$.

Remark 3.1. In the literature of the Newmark scheme, most of the time, the parameters used are (β, γ), but to avoid confusion with Nitsche's parameter γ_N, γ is replaced by ζ here.

4. Energy estimates and stability

The mechanical energy $E(t)$ is conserved for a zero external force (for continuous case only). However, this property no longer holds for the fully discrete formulations (3.1) and (3.2). Let us define the fully discrete mechanical energy associated with the solution $\mathbf{u}^{h,n}$:

$$E^{h,n} := \frac{1}{2}\left\|\rho^{\frac{1}{2}} \dot{\mathbf{u}}^{h,n}\right\|_{0,\Omega}^2 + \frac{1}{2} a(\mathbf{u}^{h,n}, \mathbf{u}^{h,n}),$$

the fully discrete modified energy:

$$\begin{aligned} E_{\Theta_N}^{h,n} &:= E^{h,n} - \frac{\Theta_N}{2}\left\|\gamma_N^{-\frac{1}{2}} \sigma_n(\mathbf{u}^{h,n})\right\|_{0,\Gamma_C}^2 + \frac{\Theta_N}{2}\left\|\gamma_N^{-\frac{1}{2}} [P_{1,\gamma_N}(\mathbf{u}^{h,n})]_{\mathbb{R}^-}\right\|_{0,\Gamma_C}^2 = E^{h,n} - \Theta_N R^{h,n} \\ &= \frac{1}{2}\left\|\rho^{\frac{1}{2}} \dot{\mathbf{u}}^{h,n}\right\|_{\Omega}^2 + \frac{1}{2} a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n}, \mathbf{u}^{h,n}) + \frac{\Theta_N}{2}\left\|\gamma_N^{-\frac{1}{2}} [P_{1,\gamma_N}(\mathbf{u}^{h,n})]_{\mathbb{R}^-}\right\|_{0,\Gamma_C}^2, \end{aligned}$$

and the fully discrete counterpart of the BFF energy:

$$E_{BFF}^{h,n} := E^{h,n} + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P_{1,\gamma_N}(\mathbf{u}^{h,n})]_{\mathbb{R}^-} - \sigma_n(\mathbf{u}^{h,n}))\right\|_{\Gamma_C}^2. \tag{4.1}$$

Note that these energies $E^{h,n}$, $E_{\Theta_N}^{h,n}$ and $E_{BFF}^{h,n}$ are the fully discrete version of the energies introduced in Section 3. The evolution of the modified energy for the θ-scheme (and Newmark scheme) and its unconditional stability for $\theta = 1$ (and $\beta = \frac{1}{2}$, $\zeta = 1$, respectively) has been proven in [30]. Here, we complete the analysis of the evolution of the BFF energy.

4.1. **Properties of discrete acceleration**

In order to simplify the notations, the following conventions are used: $P^n = P_{1,\gamma_N}(\mathbf{u}^{h,n}), \sigma_n^n = \sigma_n(\mathbf{u}^{h,n})$ for all $n \geq 0$. The following two Lemma will be used for the analysis of the boundedness of the discrete energies.

Lemma 4.1. *Assume that the mesh $\mathcal{T}^h$ is quasi-uniform and $\mathbf{u}^h$ is the solution of Problem* (2.5)*, then there exists C independent of γ_0, h and Δt, such that the following inequality holds:*

$$\left\|\rho^{\frac{1}{2}}\ddot{\mathbf{u}}^h\right\|_\Omega \leq C\frac{\gamma_0}{h^{\frac{3}{2}}}\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}.$$

Proof. This proof is a rearranging of [33, Proposition 15 and 21]. Taking $\mathbf{v}^h = \ddot{\mathbf{u}}^h$ in (2.5), we have:

$$\left\|\rho^{\frac{1}{2}}\ddot{\mathbf{u}}^h\right\|_\Omega^2 = -a_{\Theta_N,\gamma_N}(\mathbf{u}^h,\ddot{\mathbf{u}}^h) - \left(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^h)\right)_{\Gamma_C}. \tag{4.2}$$

Then, we use the continuity of $a(\,\cdot\,,\cdot\,)$, the Cauchy–Schwarz inequality, the first discrete trace inequality (2.8), and the inverse inequality (2.9) to bound the first term:

$$\begin{aligned}\left|a_{\Theta_N,\gamma_N}(\mathbf{u}^h,\ddot{\mathbf{u}}^h)\right| &\leq \left|a(\mathbf{u}^h,\ddot{\mathbf{u}}^h)\right| + |\Theta_N|\left|\left(\gamma_N^{-1}\sigma_n(\mathbf{u}^h),\sigma_n(\ddot{\mathbf{u}}^h)\right)\right)_{\Gamma_C}\right| \\ &\leq C\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_{H^1(\Omega)} + |\Theta_N|\left\|\gamma_N^{-\frac{1}{2}}\sigma_n(\mathbf{u}^h)\right\|_{\Gamma_C}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n(\ddot{\mathbf{u}}^h)\right\|_{\Gamma_C} \\ &\leq Ch^{-1}\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_\Omega + |\Theta_N|C_{tr}\gamma_0^{-1}\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_{H^1(\Omega)} \\ &\leq Ch^{-1}(1+\gamma_0^{-1})\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_\Omega.\end{aligned}$$

Similarly, for the second term, we use the Cauchy–Schwarz inequality and the boundedness of the operator P_{Θ_N,γ_N} (2.12)–(2.13):

$$\begin{aligned}\left|\left(\frac{1}{\gamma_N}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}-, P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^h)\right)_{\Gamma_C}\right| &\leq \left\|\gamma_N^{-\frac{1}{2}}[P_{1,\gamma_N}(\mathbf{u}^h)]_{\mathbb{R}^-}-\right\|_{\Gamma_C}\left\|\gamma_N^{-\frac{1}{2}}P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^h)\right\|_{\Gamma_C} \\ &\leq \left\|\gamma_N^{-\frac{1}{2}}P_{1,\gamma_N}(\mathbf{u}^h)\right\|_{\Gamma_C}\left\|\gamma_N^{-\frac{1}{2}}P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^h)\right\|_{\Gamma_C} \\ &\leq C\frac{\gamma_0}{h^{\frac{3}{2}}}\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_\Omega.\end{aligned}$$

We combine the previous inequalities to deduce the following one:

$$\rho\left\|\ddot{\mathbf{u}}^h\right\|_\Omega^2 \leq C\left(h^{-1}+\gamma_0^{-1}h^{-1}+\gamma_0 h^{-\frac{3}{2}}\right)\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_\Omega \leq C\frac{\gamma_0}{h^{\frac{3}{2}}}\left\|\mathbf{u}^h\right\|_{H^1(\Omega)}\left\|\ddot{\mathbf{u}}^h\right\|_\Omega.$$

where we have used that γ_0 is large enough. ■

Lemma 4.2. *Assume that the mesh $\mathcal{T}^h$ is quasi-uniform and $0 \leq n < N$, $\mathbf{u}^{h,n}$ is the solution of* (3.1) (iii) *or* (3.2) (iii)*, then there exists C independent of γ_0, h and Δt, such that the following inequality holds:*

$$\begin{aligned}&\left\|\rho^{\frac{1}{2}}(\ddot{\mathbf{u}}^{h,n+1}-\ddot{\mathbf{u}}^{h,n})\right\|_\Omega \\ &\quad\leq \frac{C}{h}\left((1+\gamma_0^{-1})\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)} + \gamma_0^{\frac{1}{2}}\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}\right).\end{aligned}$$

Proof. Taking $\mathbf{v}^h = \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}$ in (3.1) (iii) or (3.2) (iii):

$$\begin{aligned}
&\left\| \rho^{\frac{1}{2}} (\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right\|_{\Omega}^2 \\
&\quad = -a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}, \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) - \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right)_{\Gamma_C} \\
&\qquad + a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n}, \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) + \left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-}, P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right)_{\Gamma_C} \\
&\quad \le \left| a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right| \\
&\qquad + \left| \left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^{n}]_{\mathbb{R}^-}), P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right)_{\Gamma_C} \right|.
\end{aligned}$$

Proceeding as in the previous Lemma (4.1), we have :

$$\left| a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right| \le Ch^{-1}(1+\gamma_0^{-1}) \left\| \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n} \right\|_{H^1(\Omega)} \left\| \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n} \right\|_{\Omega},$$

and we continue with the Cauchy–Schwarz inequality and the boundedness of the operator (2.12):

$$\begin{aligned}
&\left| \left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^{n}]_{\mathbb{R}^-}), P_{\Theta_N,\gamma_N}(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}) \right)_{\Gamma_C} \right| \\
&\qquad \le C \frac{\gamma_0^{\frac{1}{2}}}{h} \left\| \gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^{n}]_{\mathbb{R}^-}) \right\|_{\Gamma_C} \left\| \ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n} \right\|_{\Omega}.
\end{aligned}$$

Combining these two terms concludes the proof. ∎

4.2. Stability of the θ-scheme

The evolution of the modified energy for the θ-scheme and its unconditional stability for $\theta = 1$ has been proven in [30]. We complete here the study of the conditional boundedness of the modified energy in the case $\theta \in (\frac{1}{2}, 1]$.

Proposition 4.3 (Boundedness of the modified energy for θ-scheme)**.** *Suppose that* $\mathbf{L}^n = \mathbf{0}$, $0 < n \le N$, $\Theta_N = 1$, $\theta \in (\frac{1}{2}, 1]$, *and* γ_0 *large enough such that the following condition holds*

$$0 < C \frac{(1-\theta)\theta^4}{\theta - \frac{1}{2}} \frac{\Delta t^4}{\rho^2 h^4} \le \gamma_0^{-2} \le C_{wp}^2 \left(1 + \frac{\rho h^2}{\Delta t^2} \right)^2, \tag{4.3}$$

with $C > 0$ *independent of* θ, γ_0, h *and* Δt*. Then, the modified energy* $E_{\Theta_N}^{h,n}$ *remains bounded, i.e., for all* $0 \le n < N$*, there holds*

$$E_{\Theta_N}^{h,n+1} \le \left(1 + \sqrt{\frac{C(\theta - \frac{1}{2})(1-\theta)}{\theta^4}} \left(1 + \sqrt{\frac{(\theta - \frac{1}{2})(1-\theta)^3}{C} \frac{\gamma_0}{h}} \right) \right) E_{\Theta_N}^{h,n}. \tag{4.4}$$

Proof. According to (3.1), we have

$$\dot{\mathbf{u}}^{h,n+1} + \dot{\mathbf{u}}^{h,n} = \frac{2}{\Delta t}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + (1 - 2\theta)(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}), \tag{4.5}$$

and

$$\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n} = \theta \Delta t \ddot{\mathbf{u}}^{h,n+1} + (1-\theta)\Delta t \ddot{\mathbf{u}}^{h,n}. \tag{4.6}$$

Firstly, using the definition of $E^{h,n}_{\Theta_N}$ and (4.5), we get:

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&\quad = \frac{1}{2}\Big(\rho(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}), \dot{\mathbf{u}}^{h,n+1} + \dot{\mathbf{u}}^{h,n}\Big)_\Omega + \frac{1}{2} a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} + \mathbf{u}^{h,n}) \\
&\qquad + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} - \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} \\
&\quad = \frac{1}{\Delta t}\Big(\rho(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}), \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\Big)_\Omega + \left(\frac{1}{2} - \theta\right)\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right\|^2_\Omega \\
&\qquad + \frac{1}{2} a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} + \mathbf{u}^{h,n}) + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} - \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}.
\end{aligned}$$

Then, we use (4.6) with (3.1) (iii) and $P_{\Theta_N,\gamma_N}(\mathbf{v}^{h,n}) = P^n$ for $\Theta_N = 1$, to have

$$\begin{aligned}
&\Big(\rho(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}), \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\Big)_\Omega \\
&\quad = \theta\Big(\rho(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}), \Delta t \ddot{\mathbf{u}}^{h,n+1}\Big)_\Omega + (1-\theta)\Big(\rho(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}), \Delta t \ddot{\mathbf{u}}^{h,n}\Big)_\Omega \\
&\quad = -\Delta t \theta\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, P^{n+1} - P^n)\right)_{\Gamma_C}\right) \\
&\qquad - \Delta t(1-\theta)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-}, P^{n+1} - P^n)\right)_{\Gamma_C}\right).
\end{aligned}$$

So,

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&\quad = -\theta\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, P^{n+1} - P^n)\right)_{\Gamma_C}\right) \\
&\qquad - (1-\theta)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-}, P^{n+1} - P^n)\right)_{\Gamma_C}\right) \\
&\qquad + \left(\frac{1}{2} - \theta\right)\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right\|^2_\Omega \\
&\qquad + \frac{1}{2} a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} + \mathbf{u}^{h,n}) + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} - \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C},
\end{aligned}$$

and

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&\quad = \left(\frac{1}{2} - \theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right\|^2_\Omega + a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\right) \\
&\qquad - \theta\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}), P^{n+1} - P^n\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, P^{n+1} - P^n\right)_{\Gamma_C} \\
&\qquad + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} - \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}.
\end{aligned}$$

Since, we have $P^n[P^n]_{\mathbb{R}^-} = [P^n]^2_{\mathbb{R}^-}$, $P^n = [P^n]_{\mathbb{R}^-} + [P^n]_{\mathbb{R}^+}$, and $[P^n]_{\mathbb{R}^-}[P^n]_{\mathbb{R}^+} = 0$, we get finally

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&= \left(\frac{1}{2} - \theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right\|^2_{\Omega} + a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\right) \\
&\quad - \theta\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C} - \theta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\right)_{\Gamma_C} \\
&\quad - \theta\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, [P^{n+1}]_{\mathbb{R}^+}\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, [P^{n+1}]_{\mathbb{R}^-}\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, [P^{n+1}]_{\mathbb{R}^+}\right)_{\Gamma_C} \\
&\quad + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^n]_{\mathbb{R}^-}\right\|^2_{\Gamma_C} \\
&= \left(\frac{1}{2} - \theta\right)\Bigg(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right\|^2_{\Omega} \\
&\quad + a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C}\Bigg) \\
&\quad + \underbrace{\theta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\right)_{\Gamma_C}}_{\leq 0} + (\theta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}.
\end{aligned}$$

The last term is bounded using the inequality (2.6), the boundedness of operator P (Lemma 2.6) and the definition of the θ-scheme (3.1):

$$\begin{aligned}
&(\theta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\qquad \leq \frac{(1-\theta)}{4}\left\|\gamma_N^{-\frac{1}{2}}(P^{n+1} - P^n)\right\|^2_{\Gamma_C} \\
&\qquad \leq C(1-\theta)\frac{\gamma_0}{h^2}\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|^2_{\Omega} \\
&\qquad = C(1-\theta)\frac{\gamma_0 \Delta t^2}{h^2}\left\|\dot{\mathbf{u}}^{h,n} + \Delta t\theta(1-\theta)\ddot{\mathbf{u}}^{h,n} + \Delta t\theta^2(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n})\right\|^2_{\Omega} \\
&\qquad \leq C(1-\theta)\frac{\gamma_0 \Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \Delta t^2\theta^2(1-\theta)^2\left\|\ddot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \Delta t^2\theta^4\left\|\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}\right\|^2_{\Omega}\right).
\end{aligned}$$

Now, using Lemma 4.1, Lemma 4.2 and the triangular inequality, we have

$$\begin{aligned}
&(\theta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\leq C(1-\theta)\frac{\gamma_0 \Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \frac{\gamma_0^2 \Delta t^2 \theta^2 (1-\theta)^2}{\rho^2 h^3}\left\|\mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)}\right) \\
&\quad + C(1-\theta)\theta^4\frac{\gamma_0 \Delta t^4}{\rho^2 h^4}\left((1 + \gamma_0^{-1})^2\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)} + \gamma_0\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C}\right).
\end{aligned}$$

Hence, using previous inequality, Lemma (2.7) and condition (4.3) for γ_0 large enough, we have

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&\quad\le \left(\frac{1}{2}-\theta\right)\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|^2_{\Omega} \\
&\qquad + C(1-\theta)\frac{\gamma_0\Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \frac{\gamma_0^2\Delta t^2\theta^2(1-\theta)^2}{\rho^2 h^3}\left\|\mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)}\right) \\
&\qquad + \left(\left(\frac{1}{2}-\theta\right) + C\gamma_0^2(1-\theta)\theta^4\frac{\Delta t^4}{\rho^2 h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C} \\
&\qquad + \gamma_0^{-1}\left(\left(\frac{1}{2}-\theta\right)(C_a\gamma_0 - C_{tr}) + C(1+\gamma_0^{-1})^2\gamma_0^2(1-\theta)\theta^4\frac{\Delta t^4}{\rho^2 h^4}\right)\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)} \\
&\quad\le C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \frac{\gamma_0^2\Delta t^2\theta^2(1-\theta)^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)}\right).
\end{aligned}$$

Then, the modified energy is bounded using its definition

$$\begin{aligned}
E^{h,n+1}_{\Theta_N} &\le E^{h,n}_{\Theta_N} + C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|^2_{\Omega} + \frac{\gamma_0^2\theta^2(1-\theta)^2\Delta t^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|^2_{H^1(\Omega)}\right) \\
&\le \left(1 + C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1+\frac{\gamma_0^2\theta^2(1-\theta)^2\Delta t^2}{\rho h^3}\right)\right)E^{h,n}_{\Theta_N},
\end{aligned}$$

and we conclude the proof using again (4.3). ■

Remark 4.4. The left part of the condition (4.3) can be seen as a CFL condition since it implies an upper bound to $\frac{\Delta t}{\rho^{\frac{1}{2}}h}$. The right part is the well-posedness condition.

Remark 4.5 (Unconditional stability)**.** From (4.4), we conclude that we have no unconditional stability except for $\theta = 1$ (see above Corollary) however taking γ_0 small enough limits the augmentation of the modified energy between two time-steps (though it needs to be large enough to ensure well-posedness).

Remark 4.6 (h-dependency)**.** The constant in (4.4) depends on h, hence the time discretization error is influenced by the space discretization. This dependency comes from Lemma 4.1 which has a factor $h^{\frac{3}{2}}$. This is suboptimal and improving this result remains an open question.

The unconditional stability of the backward Euler scheme (θ-scheme with $\theta = 1$) proven in [30] is recovered here

Corollary 4.7 (Unconditional stability of backward Euler scheme)**.** *Assume that* $\mathbf{L}^n = \mathbf{0}$, $0 \le n < N$, *and* γ_0 *is large enough (to ensure well-posedness). Then, with* $\theta = 1$ *and* $\Theta_N = 1$, *the solution of Problem* (3.1) *is unconditionally stable in the sense of the modified energy, i.e.,*

$$E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \le 0, \quad \forall\, n > 0.$$

Proof. It follows directly from

$$\left(1+\sqrt{\frac{C(\theta-\frac{1}{2})(1-\theta)}{\theta^4}}\left(1+\sqrt{\frac{(\theta-\frac{1}{2})(1-\theta)^3}{C}\frac{\gamma_0}{h}}\right)\right) = 1.$$ ■

Then, we state the evolution of the BFF energy for θ-scheme.

Proposition 4.8 (Evolution of the BFF energy for θ-scheme)**.** *Suppose that* $\mathbf{L}^n = \mathbf{0}$ *and that Problem* (3.1) *is well-posed. The following identity holds for all* $n \geq 0$*:*

$$
\begin{aligned}
E^{h,n+1}_{BFF} - E^{h,n}_{BFF} = {} & \Big(\frac{1}{2} - \theta\Big)\Big(\Big\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\Big\|^2_{\Omega} + a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\Big) \\
& + \Big(\frac{1}{2} - \theta\Big)\Big\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\Big\|^2_{\Gamma_C} \\
& + \Big(\frac{1}{2} + \Theta_N\theta\Big)\Big\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1} - \sigma_n^n)\Big\|^2_{\Gamma_C} + (\Theta_N + 1)\Big(\frac{1}{\gamma_N}\sigma_n^n, (\sigma_n^{n+1} - \sigma_n^n)\Big)_{\Gamma_C} \\
& + (1 - \Theta_N)\Big(\frac{1}{\gamma_N}(\theta[P^{n+1}]_{\mathbb{R}^-} + (1-\theta)([P^n]_{\mathbb{R}^-}), (\sigma_n^{n+1} - \sigma_n^n)\Big)_{\Gamma_C} \\
& + \theta\Big(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\Big)_{\Gamma_C} + (\theta - 1)\Big(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\Big)_{\Gamma_C} \\
& - \Big(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, \sigma_n^{n+1}\Big)_{\Gamma_C} + \Big(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, \sigma_n^n\Big)_{\Gamma_C}.
\end{aligned}
$$

Proof. The definition of $E^{h,n}_{BFF}$ gives:

$$
\begin{aligned}
E^{h,n+1}_{BFF} - E^{h,n}_{BFF} = {} & \frac{1}{2}\Big(\rho(\dot{\mathbf{u}}^{h,n+1} + \dot{\mathbf{u}}^{h,n}), \dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}\Big)_{\Omega} + \frac{1}{2}a(\mathbf{u}^{h,n+1} + \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) \\
& + \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - \sigma_n^{n+1})\Big\|^2_{\Gamma_C} - \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}([P^n]_{\mathbb{R}^-} - \sigma_n^n)\Big\|^2_{\Gamma_C}.
\end{aligned}
$$

Then, we use (4.5), (4.6), and the fact that $P_{\Theta_N,\gamma_N}(\mathbf{v}^h) = P_{1,\gamma_N}(\mathbf{v}^h) + (\Theta_N - 1)\sigma_n(\mathbf{v}^h)$:

$$
\begin{aligned}
E^{h,n+1}_{BFF} - E^{h,n}_{BFF} = {} & \frac{1}{\Delta t}\Big(\rho(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}), \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\Big)_{\Omega} + \Big(\frac{1}{2} - \theta\Big)\Big\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\Big\|^2_{\Omega} \\
& + \frac{1}{2}a(\mathbf{u}^{h,n+1} + \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) \\
& + \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\Big\|^2_{\Gamma_C} - \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}\sigma_n^n\Big\|^2_{\Gamma_C} + \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\Big\|^2_{\Gamma_C} \\
& - \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}[P^n]_{\mathbb{R}^-}\Big\|^2_{\Gamma_C} - \Big(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, \sigma_n^{n+1}\Big)_{\Gamma_C} + \Big(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, \sigma_n^n\Big)_{\Gamma_C} \\
= {} & \Big(\frac{1}{2} - \theta\Big)\Big(\Big\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\Big\|^2_{\Omega} + a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\Big) \\
& + \Theta_N\Big(\frac{1}{\gamma_N}(\theta\sigma_n^{n+1} + (1-\theta)\sigma_n^n), (\sigma_n^{n+1} - \sigma_n^n)\Big)_{\Gamma_C} \\
& + \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\Big\|^2_{\Gamma_C} - \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}\sigma_n^n\Big\|^2_{\Gamma_C} \\
& - \Big(\frac{1}{\gamma_N}(\theta[P^{n+1}]_{\mathbb{R}^-} + (1-\theta)[P^n]_{\mathbb{R}^-}), P_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\Big)_{\Gamma_C} \\
& + \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\Big\|^2_{\Gamma_C} - \frac{1}{2}\Big\|\gamma_N^{-\frac{1}{2}}[P^n]_{\mathbb{R}^-}\Big\|^2_{\Gamma_C} \\
& - \Big(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, \sigma_n^{n+1}\Big)_{\Gamma_C} + \Big(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, \sigma_n^n\Big)_{\Gamma_C}
\end{aligned}
$$

$$= \left(\frac{1}{2}-\theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|_{\Omega}^{2} + a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)$$
$$+ \left(\frac{1}{2}-\theta\right)\left(\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2}\right)$$
$$+ \left(\frac{1}{2}+\Theta_N\theta\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^{n})\right\|_{\Gamma_C}^{2} + (\Theta_N+1)\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}$$
$$- (1-\Theta_N)\left(\frac{1}{\gamma_N}(\theta[P^{n+1}]_{\mathbb{R}^-}+(1-\theta)[P^n]_{\mathbb{R}^-}),(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}$$
$$+ \theta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},[P^n]_{\mathbb{R}^+}\right)_{\Gamma_C} + (\theta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}$$
$$- \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C} + \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},\sigma_n^n\right)_{\Gamma_C}. \qquad \blacksquare$$

For the symmetric Nitsche's method ($\Theta_N = 1$), we state a boundedness result for the θ-scheme.

Proposition 4.9 (Boundedness of the BFF energy for θ-scheme)**.** *Suppose that* $\mathbf{L}^n = \mathbf{0}, 0 \le n < N$, $\Theta_N = 1$, $\theta \in (\frac{1}{2}, 1]$, *and* γ_0 *large enough such that the following condition holds*

$$0 < C\frac{2(1-\theta)\theta^4}{\theta-\frac{1}{2}}\frac{\Delta t^4}{\rho^2 h^4} \le \gamma_0^{-2} \le C_{wp}^2\left(1+\frac{\rho h^2}{\Delta t^2}\right)^2, \tag{4.7}$$

with $C > 0$ *independent of* θ, Θ_N, γ_0, h *and* Δt. *Then,* E_{BFF}^h *remains bounded, i.e., for all* $0 \le n < N$,

$$E_{BFF}^{h,n+1} \le \left(1+C\left(1+\left(\frac{\theta+\frac{3}{2}}{2\theta-1}\right)\gamma_0^{-1}+\sqrt{\frac{(\theta-\frac{1}{2})(1-\theta)}{2C\theta^4}}\left(1+\sqrt{\frac{(\theta-\frac{1}{2})(1-\theta)^3}{2C}\frac{\gamma_0}{h}}\right)\right)\right)E_{BFF}^{h,n}. \tag{4.8}$$

Proof. In what follows, we denote $C > 0$ a generic constant that can change at each occurrence. According to Proposition 4.8, we have

$$E_{BFF}^{h,n+1} - E_{BFF}^{h,n}$$
$$= \left(\frac{1}{2}-\theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|_{\Omega}^{2} + a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)$$
$$+ \left(\frac{1}{2}-\theta\right)\left(\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2}\right)*$$
$$+ \left(\frac{1}{2}+\theta\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^{n})\right\|_{\Gamma_C}^{2} + \theta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},[P^n]_{\mathbb{R}^+}\right)_{\Gamma_C}$$
$$+ (\theta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}$$
$$+ 2\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C} + \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},\sigma_n^n\right)_{\Gamma_C}$$

and thus

$$\begin{aligned}
&E_{BFF}^{h,n+1} - E_{BFF}^{h,n} \\
&\quad \le \left(\frac{1}{2}-\theta\right)\left(\left\|\rho^{\frac12}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|_\Omega^2 + a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right) \\
&\qquad + \left(\frac{1}{2}-\theta\right)\left(\left\|\gamma_N^{-\frac12}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right) \\
&\qquad + \left(\frac{1}{2}+\theta\right)\left\|\gamma_N^{-\frac12}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2 + (\theta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\qquad \underbrace{-\left(\frac{1}{\gamma_N}\sigma_n^n,([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right)_{\Gamma_C}}_{I_1} \underbrace{-\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-}),(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}}_{I_2} \\
&\qquad \underbrace{-\left(\frac{1}{\gamma_N}([P^n]_{\mathbb{R}^-}-\sigma_n^n),(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}}_{I_3} + \underbrace{\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}}_{I_4}. \qquad (4.9)
\end{aligned}$$

where we have used that $\theta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},[P^n]_{\mathbb{R}^+}\right)_{\Gamma_C} \ge 0$. Then, we use Young's inequality to bound the module of the last four terms, with positive constants C_1 and C_2:

$$\begin{aligned}
|I_1| &\le \frac{C_1}{2}\left\|\gamma_N^{-\frac12}\sigma_n^n\right\|_{\Gamma_C}^2 + \frac{1}{2C_1}\left\|\gamma_N^{-\frac12}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2 \\
|I_2| &\le \frac{C_2}{2}\left\|\gamma_N^{-\frac12}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2 + \frac{1}{2C_2}\left\|\gamma_N^{-\frac12}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2, \\
|I_3| &\le \frac{1}{2}\left\|\gamma_N^{-\frac12}([P^n]_{\mathbb{R}^-}-\sigma_n^n)\right\|_{\Gamma_C}^2 + \frac{1}{2}\left\|\gamma_N^{-\frac12}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2, \\
|I_4| &\le \frac{1}{2}\left\|\gamma_N^{-\frac12}\sigma_n^n\right\|_{\Gamma_C}^2 + \frac{1}{2}\left\|\gamma_N^{-\frac12}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2.
\end{aligned} \qquad (4.10)$$

Then, we continue with the inequality (2.6), the boundedness of operator P_{Θ_N,γ_N} (Lemma 2.6), the definition of the θ-scheme (3.1) and the triangular inequality:

$$\begin{aligned}
&(\theta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\quad \le \frac{(1-\theta)}{4}\left\|\gamma_N^{-\frac12}(P^{n+1}-P^n)\right\|_{\Gamma_C}^2 \\
&\quad \le C(1-\theta)\frac{\gamma_0}{h^2}\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_\Omega^2 \\
&\quad = C(1-\theta)\frac{\gamma_0\Delta t^2}{h^2}\left\|\dot{\mathbf{u}}^{h,n}+\Delta t\theta(1-\theta)\ddot{\mathbf{u}}^{h,n}+\Delta t\theta^2(\ddot{\mathbf{u}}^{h,n+1}-\ddot{\mathbf{u}}^{h,n})\right\|_\Omega^2 \\
&\quad \le C(1-\theta)\frac{\gamma_0\Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \Delta t^2\theta^2(1-\theta)^2\left\|\ddot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \Delta t^2\theta^4\left\|\ddot{\mathbf{u}}^{h,n+1}-\ddot{\mathbf{u}}^{h,n}\right\|_\Omega^2\right).
\end{aligned}$$

Now, using Lemma 4.1, Lemma 4.2 and the triangular inequality, we have

$$\begin{aligned}
&(\theta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\quad \le C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac12}\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2\theta^2(1-\theta)^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right) \\
&\qquad + C(1-\theta)\theta^4\frac{\gamma_0\Delta t^4}{\rho^2h^4}\left((1+\gamma_0^{-1})^2\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2 + \gamma_0\left\|\gamma_N^{-\frac12}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right). \qquad (4.11)
\end{aligned}$$

Then, combining (4.9), (4.10), (4.11) and using the first discrete trace inequality (2.8), we get

$$
\begin{aligned}
&E_{BFF}^{h,n+1} - E_{BFF}^{h,n} \\
&\leq \left(\frac{1}{2}-\theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|_{\Omega}^{2} + a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right) \\
&\quad + \left(\frac{1}{2}-\theta+\frac{1}{2C_1}+\frac{1}{2C_2}+C(1-\theta)\theta^4\frac{\gamma_0^2\Delta t^4}{\rho^2h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2} \\
&\quad + \left(\frac{3}{2}+\theta+\frac{C_2}{2}\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^{2} + C(1+\gamma_0^{-1})^2(1-\theta)\theta^4\frac{\gamma_0\Delta t^4}{\rho^2h^4}\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2} \\
&\quad + C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_{\Omega}^{2} + \frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P^n]_{\mathbb{R}^-}-\sigma_n^n)\right\|_{\Gamma_C}^{2} \\
&\quad + \left(C(1-\theta)^3\theta^2\frac{\gamma_0^3\Delta t^4}{\rho^2h^5}+(\frac{1}{2}+\frac{C_1}{2})C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2}. \qquad (4.12)
\end{aligned}
$$

Now, the first three lines of (4.12) are bounded using the fact that $\sigma_n^{n+1}-\sigma_n^n=\sigma_n(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})$, the first discrete trace inequality (2.8) and the coercivity of the bilinear form $a(\,\cdot\,,\cdot\,)$:

$$
\begin{aligned}
&\left(\frac{1}{2}-\theta\right)\left(\left\|\rho^{\frac{1}{2}}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right\|_{\Omega}^{2} + a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right) \\
&\quad + \left(\frac{1}{2}-\theta+\frac{1}{2C_1}+\frac{1}{2C_2}+C(1-\theta)\theta^4\frac{\gamma_0^2\Delta t^4}{\rho^2h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2} \\
&\quad + \left(\frac{3}{2}+\theta+\frac{C_2}{2}\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^{2} + C(1+\gamma_0^{-1})^2(1-\theta)\theta^4\frac{\gamma_0\Delta t^4}{\rho^2h^4}\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2} \\
&\leq \left(\left(\frac{1}{2}-\theta\right)C_a+\left(\frac{3}{2}+\theta+\frac{C_2}{2}\right)C_{tr}\gamma_0^{-1}+C(1+\gamma_0^{-1})^2(1-\theta)\theta^4\frac{\gamma_0\Delta t^4}{\rho^2h^4}\right)\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2} \\
&\quad + \left(\frac{1}{2}-\theta+\frac{1}{2C_1}+\frac{1}{2C_2}+C(1-\theta)\theta^4\frac{\gamma_0^2\Delta t^4}{\rho^2h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2} \\
&= \gamma_0^{-1}\left(C(1+\gamma_0^{-1})^2\gamma_0^2(1-\theta)\theta^4\frac{\Delta t^4}{\rho^2h^4}-(\theta-\frac{1}{2})\left(C_a\gamma_0-C_{tr}\left(1+\frac{1+2(\theta-\frac{1}{2})}{(\theta-\frac{1}{2})^2}\right)\right)\right) \\
&\qquad \times\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2} \\
&\quad + \left(C\gamma_0^2(1-\theta)\theta^4\frac{\Delta t^4}{\rho^2h^4}-\frac{1}{2}\left(\theta-\frac{1}{2}\right)\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^{2} \leq 0.
\end{aligned}
$$

where we have taken $\frac{1}{2C_1}=\frac{1}{2C_2}=\frac{1}{4}(\theta-\frac{1}{2})$ and used the condition (4.7) to conclude. It remains to bound the last two lines of (4.12) as follows using the coercivity of a_{Θ_N,γ_N} (Lemma 2.7) and the definition of $E_{BFF}^{h,n}$

$$
\begin{aligned}
&C(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_{\Omega}^{2}+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P^n]_{\mathbb{R}^-}-\sigma_n^n)\right\|_{\Gamma_C}^{2} \\
&\quad + \left(C(1-\theta)\theta^2\frac{\gamma_0^3\Delta t^4}{\rho^2h^5}+(\frac{1}{2}+\frac{C_1}{2})C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^{2} \\
&\qquad \leq C\left(1+\left(\frac{\theta+\frac{3}{2}}{2\theta-1}\right)\gamma_0^{-1}+(1-\theta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1+\theta^2(1-\theta)^2\frac{\gamma_0^2\Delta t^2}{\rho h^3}\right)\right)E_{BFF}^{h,n}.
\end{aligned}
$$

Hence, combining previous terms gives :

$$E^{h,n+1}_{BFF} \leq \left(1 + C\left(1 + \left(\frac{\theta + \frac{3}{2}}{2\theta - 1}\right)\gamma_0^{-1} + (1-\theta)\frac{\gamma_0 \Delta t^2}{\rho h^2}\left(1 + \theta^2(1-\theta)^2\frac{\gamma_0^2 \Delta t^2}{\rho h^3}\right)\right)\right)E^{h,n}_{BFF}.$$

We conclude using again (4.8). ■

Remark 4.10 (backward Euler scheme)**.** Contrary to the modified energy, we did not manage to prove that the BFF energy yields an unconditionally stable method for $\theta = 1$. However, a larger value of γ_0 increases the stability since the constant becomes $\left(1 + C\left(1 + \frac{7}{2}\gamma_0^{-1}\right)\right)$.

4.3. Stability of Newmark scheme

We state the conditional boundedness of the discrete energy for the Newmark scheme.

Proposition 4.11 (Boundedness of the modified energy for Newmark scheme)**.** *Suppose that* $\mathbf{L}^n = \mathbf{0}, 0 \leq n < N$, $\Theta_N = 1$, $\zeta = 2\beta, \zeta \in (\frac{1}{2}, 1]$, *and* γ_0 *large enough such that the following condition holds*

$$0 < C\frac{(1-\zeta)\gamma^2}{4\zeta - 2}\frac{\Delta t^4}{\rho^2 h^4} \leq \gamma_0^{-2} \leq C_{wp}^2\left(1 + \frac{2\rho h^2}{\zeta \Delta t^2}\right)^2, \tag{4.13}$$

with $C > 0$ *independent of* ζ, γ_0, h *and* Δt. *Then, the modified energy* $E^{h,n}_{\Theta_N}$ *remains bounded, i.e., for all* $0 \leq n < N$ *such that*

$$E^{h,n+1}_{\Theta_N} \leq \left(1 + C(1-\zeta)\frac{\gamma_0 \Delta t^2}{\rho h^2}\left(1 + \frac{\gamma_0^2 \Delta t^2}{\rho h^3}\right)\right)E^{h,n}_{\Theta_N}. \tag{4.14}$$

Proof. Using the definition of the modified energy $E^{h,n}_{\Theta_N}$, we get

$$\begin{aligned}
&E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \\
&\quad = \left(\frac{1}{2} - \zeta\right)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C}\right) \\
&\qquad + \zeta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\right)_{\Gamma_C} + (\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\quad \leq \left(\frac{1}{2} - \zeta\right)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C}\right) \\
&\qquad + (\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}.
\end{aligned}$$

The last term can be bounded using the inequality (2.6), the boundedness of operator P_{Θ_N,γ_N} (Lemma 2.6) and the scheme (3.2) as follows

$$\begin{aligned}
&(\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\quad \leq \frac{(1-\zeta)}{4}\left\|\gamma_N^{-\frac{1}{2}}(P^{n+1} - P^n)\right\|^2_{\Gamma_C} \\
&\quad \leq C(1-\zeta)\frac{\gamma_0}{h^2}\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|^2_{\Omega}
\end{aligned}$$

$$= C(1-\zeta)\frac{\gamma_0\Delta t^2}{h^2}\left\|\dot{\mathbf{u}}^{h,n} + \frac{\Delta t}{2}\ddot{\mathbf{u}}^{h,n} + \Delta t\frac{\zeta}{2}(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n})\right\|_\Omega^2$$
$$\leq C(1-\zeta)\frac{\gamma_0\Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\Delta t^2}{4}\left\|\ddot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \Delta t^2\frac{\zeta^2}{4}\left\|\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}\right\|_\Omega^2\right). \quad (4.15)$$

Using Lemma 4.1, Lemma 4.2 and the triangular inequality, we get:

$$(\zeta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}$$
$$\leq C(1-\zeta)\frac{\gamma_0\Delta t^2}{h^2}\left(\left\|\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2}{\rho^2 h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right)$$
$$+ C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0\Delta t^4}{\rho^2 h^4}\left((1+\gamma_0^{-1})^2\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2 + \gamma_0\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right). \quad (4.16)$$

The evolution of the modified energy becomes then

$$E_{\Theta_N}^{h,n+1} - E_{\Theta_N}^{h,n}$$
$$\leq \left(\frac{1}{2}-\zeta\right)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right)$$
$$+ C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right)$$
$$+ C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0\Delta t^4}{\rho^2 h^4}\left((1+\gamma_0^{-1})^2\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2 + \gamma_0\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right).$$

Using Lemma 2.7 and the condition (4.13) on γ_0, we get:

$$E_{\Theta_N}^{h,n+1} - E_{\Theta_N}^{h,n}$$
$$\leq \gamma_0\left(C(1+\gamma_0^{-1})^2\frac{(1-\zeta)\zeta^2}{4}\frac{\Delta t^4}{\rho^2 h^4} - \left(\zeta - \frac{1}{2}\right)(C_a\gamma_0 - C_{tr})\right)\left\|\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2$$
$$+ \gamma_0^2\left(C\frac{(1-\zeta)\zeta^2}{4}\frac{\Delta t^4}{\rho^2 h^4} - \left(\zeta-\frac{1}{2}\right)\gamma_0^{-2}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2$$
$$+ C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right)$$
$$\leq C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right)$$

Finally, using the definition of the modified energy

$$E_{\Theta_N}^{h,n+1} \leq E_{\Theta_N}^{h,n} + C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_\Omega^2 + \frac{\gamma_0^2\Delta t^2}{\rho h^3}\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\right)$$
$$\leq \left(1 + C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1 + \frac{\gamma_0^2\Delta t^2}{\rho h^3}\right)\right)E_{\Theta_N}^{h,n},$$

and we conclude the proof using condition (4.11). ■

Contrary to the linear case, where we have the unconditional stability of the Newmark scheme for a range of parameters, we have in the non-linear case only the unconditional stability for the following parameter $\Theta_N = 1, \beta = \frac{1}{2}$ and $\zeta = 1$. This is proved in the following Corollary and this case coincides with [30, Corollary 3.5].

Corollary 4.12. *Assume that $\mathbf{L}^n = \mathbf{0}$, $0 \le n < N$, and γ_0 is large enough (to ensure well-posedness). Then, with $\beta = \frac{1}{2}, \zeta = 1$ and $\Theta_N = 1$, the solution of Problem (3.2) is unconditionally stable in the sense of the modified energy, i.e.,*

$$E^{h,n+1}_{\Theta_N} - E^{h,n}_{\Theta_N} \le 0, \quad \forall\, n > 0. \tag{4.17}$$

Proof. This is a direct application of Proposition 4.11 with $\gamma = 1$ and $\beta = \frac{1}{2}$ since in this is case:

$$\left(1 + \sqrt{\frac{C(4\zeta-2)(1-\zeta)}{\zeta^2}}\left(1 + \sqrt{\frac{4\zeta-2}{C(1-\zeta)\zeta^2}\frac{\gamma_0}{h}}\right)\right) = 1. \tag{4.18}$$

■

Similarly to the modified energy (Lemma 4.8), we study the evolution of BFF energy during the simulation:

Lemma 4.13 (Evolution of the BFF energy of Newmark scheme)**.** *Assume that $\mathbf{L}^n = \mathbf{0}$, $\forall\, n > 0$ and that Problem (3.2) is well-posed. Then, the following property holds for all $n > 0$:*

$$\begin{aligned}
&E^{h,n+1}_{BFF} - E^{h,n}_{BFF} \\
&= \left(\frac{1}{2} - \zeta\right) a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left(\frac{1}{2} + \zeta\Theta_N\right) \left\| \gamma_N^{-\frac{1}{2}} \sigma(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) \right\|^2_{\Gamma_C} \\
&\quad + \left(\frac{1}{2} - \zeta\right) \left\| \gamma_N^{-\frac{1}{2}} ([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}) \right\|^2_{\Gamma_C} \\
&\quad + \left(\frac{1}{\gamma_N} \zeta [P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\right)_{\Gamma_C} + (\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C} \\
&\quad + (1 + \Theta_N)\left(\frac{1}{\gamma_N}\sigma_n^n, (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, \sigma_n^{n+1}\right)_{\Gamma_C} + \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, \sigma_n^n\right)_{\Gamma_C} \\
&\quad + (1 - \Theta_N)\zeta\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}), (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C} + (1 - \Theta_N)\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C} \\
&\quad + \frac{\Delta t}{2}(2\beta - \zeta) a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n}) \\
&\quad + \frac{\Delta t}{2}(2\beta - \zeta)\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}), P_{\Theta_N,\gamma_N}(\dot{\mathbf{u}}^{h,n+1} - \dot{\mathbf{u}}^{h,n})\right)_{\Gamma_C}.
\end{aligned}$$

Proof. Let $\mathbf{v}^h \in \mathbf{V}^h$. Firstly, from equations (3.2) (i) and (ii), we deduce

$$\left(\rho(\dot{\mathbf{u}}^{h,n+1} + \dot{\mathbf{u}}^{h,n}), \mathbf{v}^h\right)_\Omega = \frac{2}{\Delta t}\left(\rho(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}), \mathbf{v}^h\right)_\Omega + \Delta t(\zeta - 2\beta)\left(\rho(\ddot{\mathbf{u}}^{h,n+1} - \ddot{\mathbf{u}}^{h,n}), \mathbf{v}^h\right)_\Omega.$$

Then, we use (3.2) (iii) (applied for time t^n and t^{n+1}) to have

$$\begin{aligned}
&\left(\rho(\dot{\mathbf{u}}^{h,n+1} + \dot{\mathbf{u}}^{h,n}), \mathbf{v}^h\right)_\Omega \\
&\quad = \frac{2}{\Delta t}\left(\rho(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}), \mathbf{v}^h\right)_\Omega + \Delta t(\zeta - 2\beta) a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{v}^h) \\
&\quad + \Delta t(\zeta - 2\beta)\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}), P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right)_{\Gamma_C}.
\end{aligned} \tag{4.19}$$

Similarly, using (3.2) (ii) and (iii), we get

$$\Big(\rho(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n}),\mathbf{v}^h\Big)_\Omega = \Delta t(\zeta-1)\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n},\mathbf{v}^h)+\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right)_{\Gamma_C}\right)$$
$$-\Delta t\zeta\left(a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1},\mathbf{v}^h)+\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},P_{\Theta_N,\gamma_N}(\mathbf{v}^h)\right)_{\Gamma_C}\right). \quad (4.20)$$

According to the definition of BFF energy and using (4.19) and (4.20), we have

$$\begin{aligned}
&E_{BFF}^{h,n+1}-E_{BFF}^{h,n}\\
&\quad=\frac{1}{2}\Big(\rho(\dot{\mathbf{u}}^{h,n+1}+\dot{\mathbf{u}}^{h,n}),\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n}\Big)_\Omega+\frac{1}{2}a(\mathbf{u}^{h,n+1}+\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\\
&\qquad+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|_{\Gamma_C}^2-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|_{\Gamma_C}^2+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\right\|_{\Gamma_C}^2-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n}\right\|_{\Gamma_C}^2\\
&\qquad-\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C}+\left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-},\sigma_n^{n}\right)_{\Gamma_C}
\end{aligned}$$

and

$$\begin{aligned}
&E_{BFF}^{h,n+1}-E_{BFF}^{h,n}\\
&\quad=(\zeta-1)\left(a(\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})-\Theta_N\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}\right)\\
&\qquad+(\zeta-1)\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},P_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}\\
&\qquad-\zeta\left(a(\mathbf{u}^{h,n+1},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})-\Theta_N\left(\frac{1}{\gamma_N}\sigma_n^{n+1},(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}\right)\\
&\qquad-\zeta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},P_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}\\
&\qquad+\underbrace{\frac{\Delta t}{2}(2\beta-\zeta)a_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})}_{T_1}\\
&\qquad+\underbrace{\frac{\Delta t}{2}(2\beta-\zeta)\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-}),P_{\Theta_N,\gamma_N}(\dot{\mathbf{u}}^{h,n+1}-\dot{\mathbf{u}}^{h,n})\right)_{\Gamma_C}}_{T_2}\\
&\qquad+\frac{1}{2}a(\mathbf{u}^{h,n+1}+\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})-\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C}+\left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-},\sigma_n^{n}\right)_{\Gamma_C}\\
&\qquad+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|_{\Gamma_C}^2-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|_{\Gamma_C}^2+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\right\|_{\Gamma_C}^2-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n}\right\|_{\Gamma_C}^2.
\end{aligned}$$

Then, we rearrange and use the relation $P_{\Theta_N,\gamma_N}(\mathbf{v}^h)=P_{1,\gamma_N}(\mathbf{v}^h)+(\Theta_N-1)\sigma_n(\mathbf{v}^h)$ to get

$$\begin{aligned}
&E_{BF}^{h,n+1}-E_{BF}^{h,n}\\
&\quad=-\zeta a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})+\zeta\Theta_N\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2\\
&\qquad-\zeta\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-}),P_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}+T_1+T_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})+\Theta_N\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}\\
&-\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},P_{\Theta_N,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}\\
&+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\right\|^2_{\Gamma_C}-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n}\right\|^2_{\Gamma_C}\\
&-\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C}+\left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-},\sigma_n^{n}\right)_{\Gamma_C}
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{BF}^{h,n+1}-E_{BF}^{h,n}\\
&=\left(\frac{1}{2}-\zeta\right)a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})+\zeta\Theta_N\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|^2_{\Gamma_C}\\
&\quad-\zeta\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-}),P_{1,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}\\
&\quad\underbrace{-\zeta(\Theta_N-1)\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-}),(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}}_{T_3}\\
&\quad+\Theta_N\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}-\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},P_{1,\gamma_N}(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)_{\Gamma_C}\\
&\quad\underbrace{-(\Theta_N-1)\left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-},(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}}_{T_4}\\
&\quad+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n+1}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}[P^{n}]_{\mathbb{R}^-}\right\|^2_{\Gamma_C}+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n+1}\right\|^2_{\Gamma_C}-\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}\sigma_n^{n}\right\|^2_{\Gamma_C}\\
&\quad-\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C}+\left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-},\sigma_n^{n}\right)_{\Gamma_C}+T_1+T_2.
\end{aligned}
$$

Finally, we conclude that

$$
\begin{aligned}
&E_{BF}^{h,n+1}-E_{BF}^{h,n}\\
&=\left(\frac{1}{2}-\zeta\right)a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})+\zeta\Theta_N\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|^2_{\Gamma_C}\\
&\quad+\left(\frac{1}{2}-\zeta\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|^2_{\Gamma_C}\\
&\quad+\zeta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},[P^n]_{\mathbb{R}^+}\right)_{\Gamma_C}+(\zeta-1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+},[P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}\\
&\quad+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|^2_{\Gamma_C}+(\Theta_N+1)\left(\frac{1}{\gamma_N}\sigma_n^n,(\sigma_n^{n+1}-\sigma_n^n)\right)_{\Gamma_C}\\
&\quad-\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-},\sigma_n^{n+1}\right)_{\Gamma_C}+\left(\frac{1}{\gamma_N}[P^{n}]_{\mathbb{R}^-},\sigma_n^{n}\right)_{\Gamma_C}\\
&\quad+T_1+T_2+T_3+T_4.
\end{aligned}
$$

■

As previously, we study to which extend the BFF energy is bounded.

Proposition 4.14 (Boundedness of the BFF energy of Newmark scheme)**.** *Suppose that*

$$\mathbf{L}^n = \mathbf{0}, \quad 0 \le n < N, \quad \Theta_N = 1, \quad \zeta = 2\beta, \quad \zeta \in \left(\frac{1}{2}, 1\right],$$

and γ_0 *large enough such that the following condition holds*

$$0 < C\frac{(1-\zeta)\gamma^2}{2\zeta - 1}\frac{\Delta t^4}{\rho^2 h^4} \le \gamma_0^{-2} \le C_{wp}^2\left(1 + \frac{2\rho h^2}{\zeta \Delta t^2}\right), \tag{4.21}$$

with $C > 0$ *independent of* ζ*,* γ_0*,* h *and* Δt*. Then, the BFF energy* $E_{BFF}^{h,n}$ *remains bounded, i.e., for all* $0 \le n \le N$*, there holds*

$$E_{BFF}^{h,n+1} \le \left(1 + C\left(1 + \left(\frac{\zeta + \frac{3}{2}}{2\zeta - 1}\right)\gamma_0^{-1} + \sqrt{\frac{C(2\zeta-1)(1-\zeta)}{\zeta^2}}\left(1 + \sqrt{\frac{2\zeta - 1}{C(1-\zeta)\zeta^2}\frac{\gamma_0}{h}}\right)\right)\right)E_{BFF}^{h,n}. \tag{4.22}$$

Proof. Using Lemma 4.13 and simplifying with assumptions on the parameters, we have:

$$\begin{aligned}
&E_{BFF}^{h,n+1} - E_{BFF}^{h,n}\\
&\quad = \left(\frac{1}{2} - \zeta\right)\left(a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right)\\
&\qquad + \left(\frac{1}{2} + \zeta\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1} - \sigma_n^n)\right\|_{\Gamma_C}^2 + \zeta\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, [P^n]_{\mathbb{R}^+}\right)_{\Gamma_C}\\
&\qquad + (\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}\\
&\qquad + 2\left(\frac{1}{\gamma_N}\sigma_n^n, (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C} - \left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^-}, \sigma_n^{n+1}\right)_{\Gamma_C} + \left(\frac{1}{\gamma_N}[P^n]_{\mathbb{R}^-}, \sigma_n^n\right)_{\Gamma_C}
\end{aligned}$$

and

$$\begin{aligned}
&E_{BFF}^{h,n+1} - E_{BFF}^{h,n}\\
&\quad \le \left(\frac{1}{2} - \zeta\right)\left(a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}) + \left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\right)\\
&\qquad + \left(\frac{1}{2} + \zeta\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1} - \sigma_n^n)\right\|_{\Gamma_C}^2 + (\zeta - 1)\left(\frac{1}{\gamma_N}[P^{n+1}]_{\mathbb{R}^+}, [P^n]_{\mathbb{R}^-}\right)_{\Gamma_C}\\
&\qquad - \underbrace{\left(\frac{1}{\gamma_N}\sigma_n^n, ([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right)_{\Gamma_C}}_{I_1} - \underbrace{\left(\frac{1}{\gamma_N}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-}), (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C}}_{I_2}\\
&\qquad - \underbrace{\left(\frac{1}{\gamma_N}([P^n]_{\mathbb{R}^-} - \sigma_n^n), (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C}}_{I_3} + \underbrace{\left(\frac{1}{\gamma_N}\sigma_n^n, (\sigma_n^{n+1} - \sigma_n^n)\right)_{\Gamma_C}}_{I_4}.
\end{aligned}$$

The terms I_1, I_2, I_3 and I_4 are treated in the same way that in Proposition 4.9, so we get

$$\begin{aligned}
&E_{BFF}^{h,n+1} - E_{BFF}^{h,n}\\
&\quad \le \left(\frac{1}{2} - \zeta\right)\left(a(\mathbf{u}^{h,n+1} - \mathbf{u}^{h,n}, \mathbf{u}^{h,n+1} - \mathbf{u}^{h,n})\right)\\
&\qquad + \left(\frac{1}{2} - \zeta + \frac{1}{2C_1} + \frac{1}{2C_2} + C\frac{(1-\zeta)\zeta}{2}\frac{\gamma_0^2\Delta t^4}{\rho^2 h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-} - [P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2
\end{aligned}$$

$$+\left(\frac{3}{2}+\zeta+\frac{C_2}{2}\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2+C\frac{(1-\zeta)\zeta}{2}\frac{\gamma_0(1+\gamma_0^{-1})^2\Delta t^4}{\rho^2h^4}\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2$$
$$+C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_{\Omega}^2+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P^n]_{\mathbb{R}^-}-\sigma_n^n)\right\|_{\Gamma_C}^2$$
$$+\left(C(1-\zeta)\frac{\gamma_0^3\Delta t^4}{\rho^2h^5}+(\frac{1}{2}+\frac{C_1}{2})C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2.$$

with $C_1>0$ and $C_2>0$. The following part of this proof is similar to Proposition 4.9. Tacking $\frac{1}{2C_1}=\frac{1}{2C_2}=\frac{1}{4}(\zeta-\frac{1}{2})$ and using the first discrete trace inequality (2.8), the coercivity of $a(\cdot,\cdot)$ and the condition (4.21), the first three lines are negative:

$$\begin{aligned}
&\left(\frac{1}{2}-\zeta\right)\left(a(\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n},\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n})\right)\\
&\quad+\left(\frac{1}{2}-\zeta+\frac{1}{2C_1}+\frac{1}{2C_2}+C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0^2\Delta t^4}{\rho^2h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\\
&\quad+\left(\frac{3}{2}+\zeta+\frac{C_2}{2}\right)\left\|\gamma_N^{-\frac{1}{2}}(\sigma_n^{n+1}-\sigma_n^n)\right\|_{\Gamma_C}^2+C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0(1+\gamma_0^{-1})^2\Delta t^4}{\rho^2h^4}\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\\
&\le\left(\left(\frac{1}{2}-\zeta\right)C_a+\left(\frac{3}{2}+\zeta+\frac{C_2}{2}\right)C_{tr}\gamma_0^{-1}+C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0(1+\gamma_0^{-1})^2\Delta t^4}{\rho^2h^4}\right)\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\\
&\quad+\left(\frac{1}{2}-\zeta+\frac{1}{2C_1}+\frac{1}{2C_2}+C\frac{(1-\zeta)\zeta^2}{4}\frac{\gamma_0^2\Delta t^4}{\rho^2h^4}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2\\
&=\gamma_0\left(C\frac{(1-\zeta)\zeta^2(1+\gamma_0^{-1})^2}{4}\frac{\Delta t^4}{\rho^2h^4}-(\zeta-\frac{1}{2})\left(C_a\gamma_0-C_{tr}\left(1+\frac{1+2(\zeta-\frac{1}{2})}{(\zeta-\frac{1}{2})^2}\right)\right)\right)\\
&\qquad\times\left\|\mathbf{u}^{h,n+1}-\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\\
&\quad+\gamma_0^2\left(C\frac{(1-\zeta)\zeta^2}{4}\frac{\Delta t^4}{\rho^2h^4}-\frac{1}{2}(\zeta-\frac{1}{2})\gamma_0^{-2}\right)\left\|\gamma_N^{-\frac{1}{2}}([P^{n+1}]_{\mathbb{R}^-}-[P^n]_{\mathbb{R}^-})\right\|_{\Gamma_C}^2<0.
\end{aligned}$$

The last two lines are always bounded as follows using the coercivty of a_{Θ_N,γ_N} and the definition of $E^{h,n}_{BFF}$

$$\begin{aligned}
&C(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left\|\rho^{\frac{1}{2}}\dot{\mathbf{u}}^{h,n}\right\|_{\Omega}^2+\frac{1}{2}\left\|\gamma_N^{-\frac{1}{2}}([P^n]_{\mathbb{R}^-}-\sigma_n^n)\right\|_{\Gamma_C}^2\\
&\quad+\left(C(1-\zeta)\frac{\gamma_0^3\Delta t^4}{\rho^2h^5}+(\frac{1}{2}+\frac{C_1}{2})C_{tr}\gamma_0^{-1}\right)\left\|\mathbf{u}^{h,n}\right\|_{H^1(\Omega)}^2\\
&\qquad\le C\left(1+\left(\frac{\zeta+\frac{3}{2}}{2\zeta-1}\right)\gamma_0^{-1}+(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1+\frac{\gamma_0^2\Delta t^2}{\rho h^3}\right)\right)E^{h,n}_{BFF}.
\end{aligned}$$

Hence, we have:

$$E^{h,n+1}_{BFF}-E^{h,n}_{BFF}\le C\left(1+\left(\frac{\zeta+\frac{3}{2}}{2\zeta-1}\right)\gamma_0^{-1}+(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1+\frac{\gamma_0^2\Delta t^2}{\rho h^3}\right)\right)E^{h,n}_{BFF}.$$

So,

$$E^{h,n+1}_{BFF}\le\left(1+C\left(1+\left(\frac{\zeta+\frac{3}{2}}{2\zeta-1}\right)\gamma_0^{-1}+(1-\zeta)\frac{\gamma_0\Delta t^2}{\rho h^2}\left(1+\frac{\gamma_0^2\Delta t^2}{\rho h^3}\right)\right)\right)E^{h,n}_{BFF}.$$

We conclude using condition (4.21). ■

5. Numerical results

This section presents the numerical simulations of an unidimensional elastic Signorini problem. Both θ-scheme and Newmark scheme are used for the benchmark with the symmetric Nitsche variant ($\Theta_N = 1$). An initial acceleration field is needed and computed from the initial displacement and velocity fields, using the relation $\ddot{\mathbf{u}}^0 = \mathbf{M}^{-1}(\mathbf{L}^0 - \mathbf{B}_N(\mathbf{u}^0))$. We consider the test-case proposed in [35, Section 5], which is described by the following dynamic problem:

$$\begin{cases} \text{Seek}\, u(x,t) : [0,1] \times (0,T) \longrightarrow \mathbb{R} \text{ s.t.} \\ \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0, \\ u(1,t) = 0, \\ u(0,t) \geq 0, \quad \dfrac{\partial u}{\partial x}(0,t) \leq 0, \quad u(0,t)\dfrac{\partial u}{\partial x}(0,t) = 0, \\ u(x,0) = \dfrac{1}{2} - \dfrac{x}{2}, \quad \dfrac{\partial u}{\partial t}(x,0) = 0. \end{cases} \tag{5.1}$$

The problem admits a unique periodic analytical solution.

5.1. θ-scheme

For the θ-scheme, we present in Figure 5.1, the simulations with different mesh sizes and a fixed ratio $\frac{\Delta t}{h} = 1$, with Nitsche's parameter $\Theta_N = 1$, $\gamma_0 = 15$, and $\theta \in \{0.51, 0.7, 1\}$. We observe important oscillations for $\theta = 0.51$, particularly with the coarse mesh ($h = 0.1$, as shown in Figure 5.1a and 5.1b), due to its lower dissipative properties compared to the other variants. We note that using $\gamma_0 = 5$ can improve the performance and reduce oscillations for $\theta = 0.51$ (see purple line in Figure 5.2). The case $\theta = 1$ gives smoother results, but the energy loss is important. The numerical results confirm that the different energies (mechanical, modified, BFF) are strictly dissipative. The case $\theta = 0.7$ corresponds to a balance between dissipation and oscillations.

Proposition 4.9 shows that the scheme with $\theta = 0.51$ is stable, but the boundedness of the discrete energy is highly dependent on the value of the parameter γ_0 (for $\frac{\Delta t}{h} = 1$ fixed). We illustrate the influence of γ_0 with $h = \Delta t = 0.025$ in Figure 5.2. It can be observed that for $\theta = 0.51$, a larger value of γ_0 introduces more oscillations during impacts, so moderate value of γ_0 is needed to prevent them and decreasing energy (γ_0 must be sufficiently large to ensure well-posedness), however for $\theta = 0.7$ or $\theta = 1$ there is almost no dependency to γ_0.

Simulations using a fixed mesh size of $h = 0.05$, $\gamma_0 = 50$, and different time-steps are plotted in Figure 5.3. For the case $\theta = 1$, we note that smaller time-steps give a better accuracy by reducing energy loss without introducing additional oscillations as expected. A zoom on the first impact in Figure 5.4 illustrates the evolution of the BFF, modified, and mechanical energies. It confirms that the modified energy remains non-increasing, contrary to the mechanical and BFF energy. Moreover the differences between the three different energies are of small magnitude.

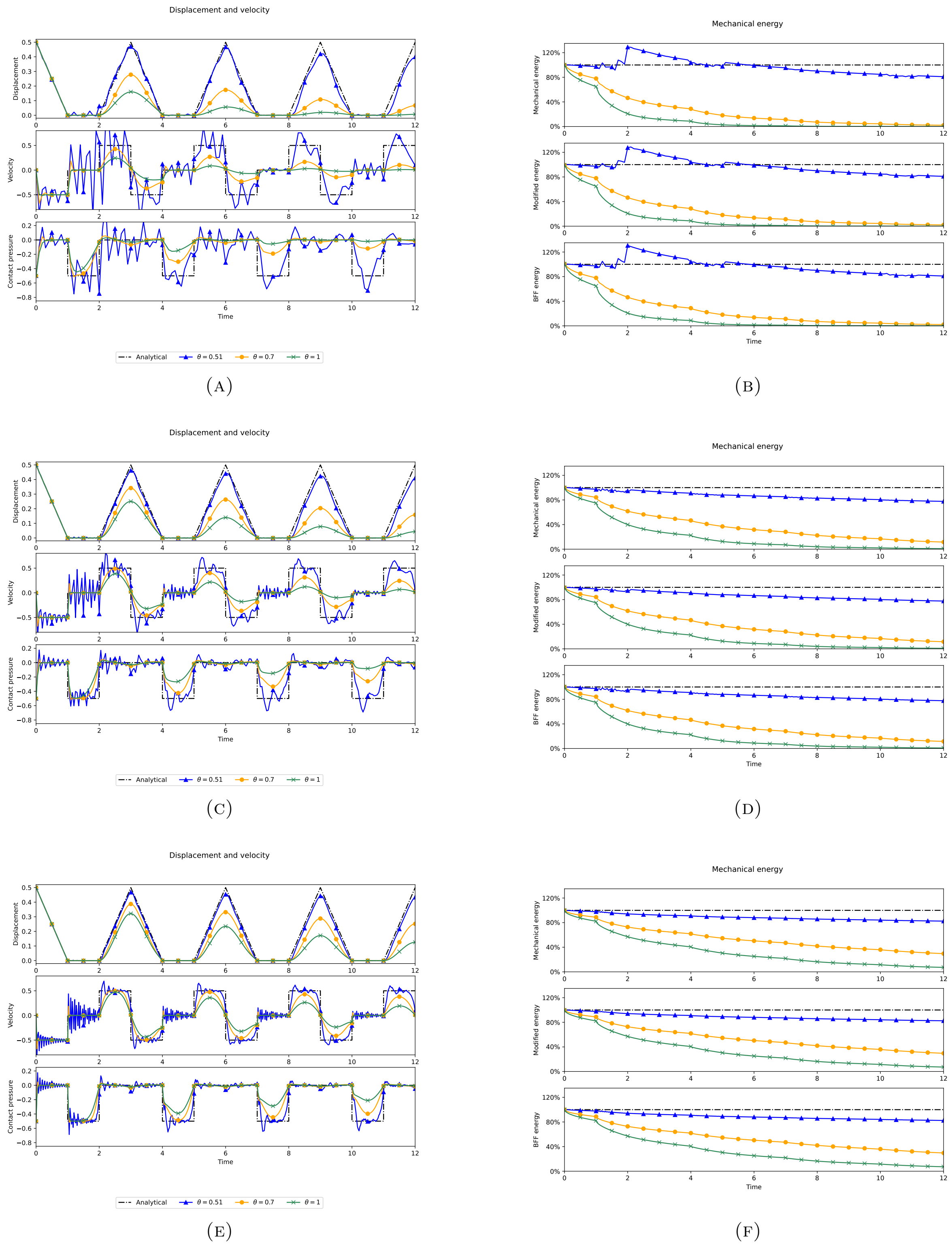


Figure 5.1. θ-scheme with $\gamma_0 = 15$ and (top) $h = \Delta t = 0.1$; (middle) $h = \Delta t = 0.05$; (bottom) $h = \Delta t = 0.025$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).

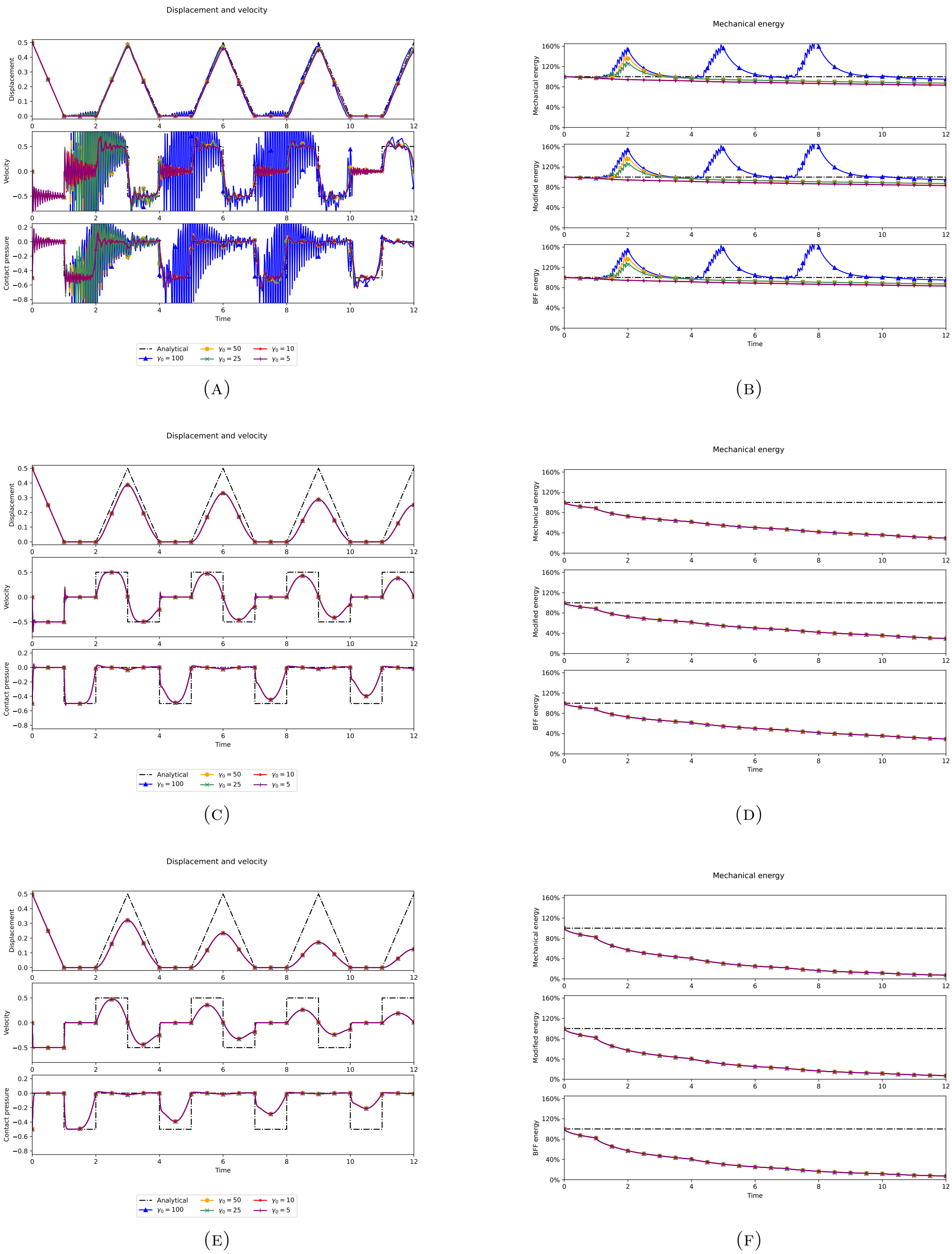
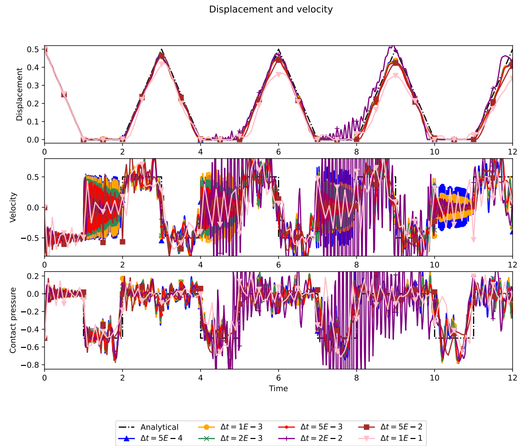
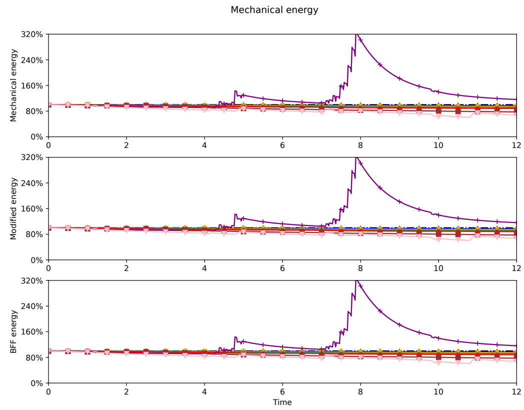


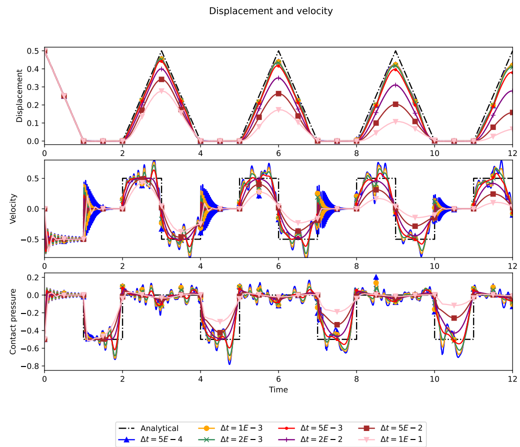
Figure 5.2. θ-scheme using different values of γ_0 ($h = \Delta t = 0.025$ fixed), with (top) $\theta = 0.51$; (middle) $\theta = 0.7$; (bottom) $\theta = 1$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).



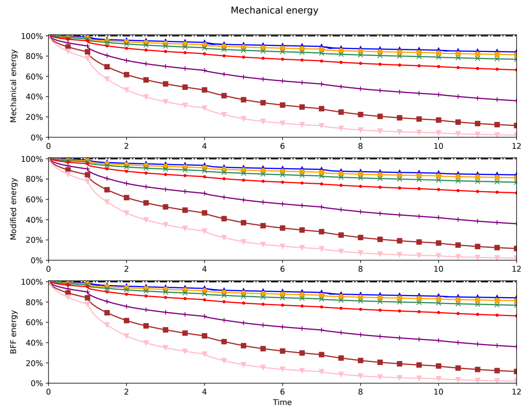
(A)



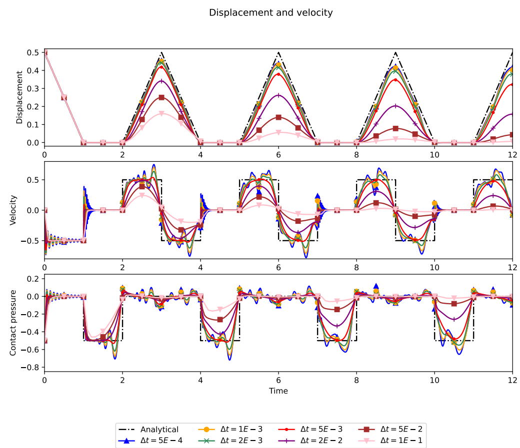
(B)



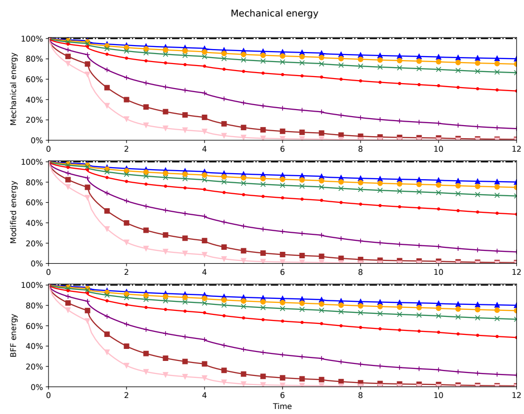
(C)



(D)

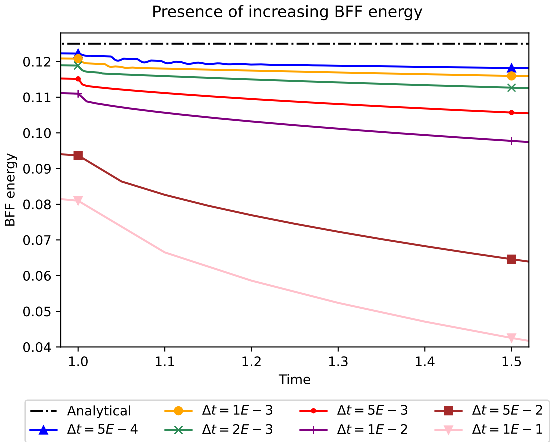


(E)

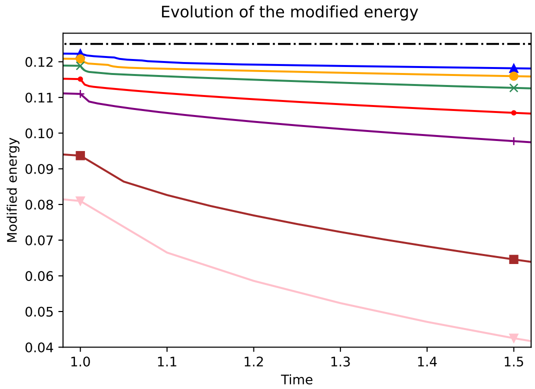


(F)

Figure 5.3. θ-scheme using $h = 0.05$, $\gamma_0 = 50$ and different Δt, with (top) $\theta = 0.51$; (middle) $\theta = 0.7$; (bottom) $\theta = 1$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).



(A)



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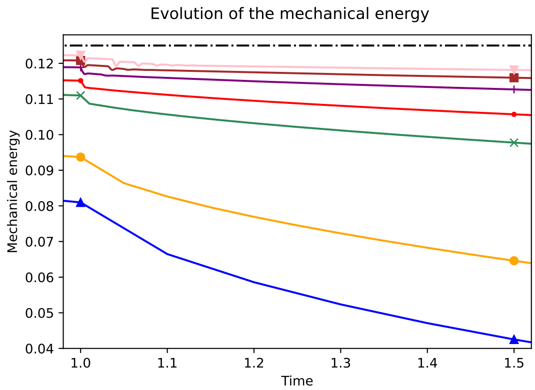


(C)

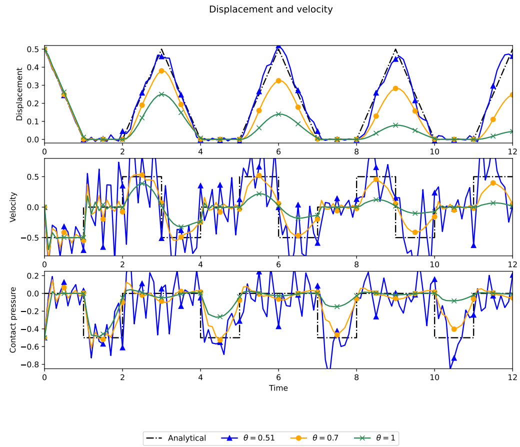
Figure 5.4. The energy evolution at the beginning of the first impact for θ-scheme with $\theta = 1$. (a) BFF energy; (b) Modified energy; (c) mechanical energy.

5.2. Newmark scheme

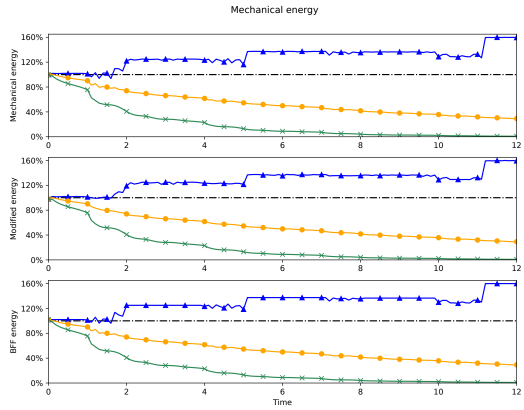
Next, we present the simulations of the Newmark scheme in Figure 5.5 for different mesh sizes and a fixed ratio $\frac{\Delta t}{h} = 1$. $\gamma_0 = 50$ is used for the highly dissipative scheme ($\beta = \frac{1}{2}, \zeta = 1$) and $\gamma_0 = 5$ for the other two schemes: the Crank–Nicolson scheme and the scheme $\beta = 0.3, \zeta = 0.6$. Even if the schemes are well-posed, energy can be amplified between two time-steps by a factor proportional to γ_0^3, resulting in increased parasitic oscillations with larger γ_0, as discussed in Proposition 4.11 and 4.14. The case of $\beta = 0.3, \zeta = 0.6$ produces smoother results than the Crank–Nicolson scheme as the parameters β and ζ help dissipate a part of the high-frequency oscillations. This represents a potential compromise between reducing parasitic oscillations and minimizing energy loss. The case of $\beta = \frac{1}{2}, \zeta = 1$ ensures unconditional stability, similar to the θ-scheme with $\theta = 1$, but it dissipates less energy than the θ-scheme.

We also show the influence of the parameter γ_0 for the Newmark scheme in Figure 5.6, with $h = \Delta t = 0.025$. Notably, the Crank–Nicolson scheme is conservative for linear elastodynamics, but with the presence of Signorini's conditions, its numerical stability is not ensured: from Figure 5.6a and 5.6b, we can observe important oscillations during the simulation, especially after several impacts. Using a dissipative scheme improves accuracy and gives smoother results (see Figure 5.6 middle and bottom). We also note that compared to the θ-scheme, the Newmark scheme with $\zeta = 2\beta$ and $\zeta \in (\frac{1}{2}, 1]$ preserves more kinetic energy, see Figure 5.2 and 5.6.

Similarly to the θ-scheme, we examine the dissipative Newmark scheme ($\beta = \frac{1}{2}, \zeta = 1$) with a fixed mesh size of $h = 0.05$ and a variable time-steps ($\gamma_0 = 50$), as shown in Figure 5.7. As with the θ-scheme, smaller time-steps result in better accuracy. Additionally, Figure 5.8 demonstrates that the modified energy does not increase, while there is an observable increase for the BFF energy at the first impact. As previously, the difference between the mechanical energy, the modified energy and the BFF energy are of small magnitude.



(A)



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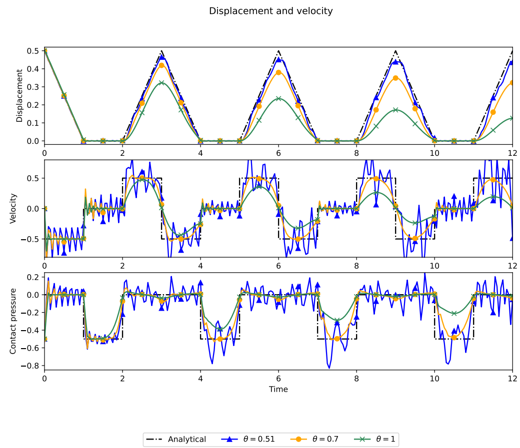
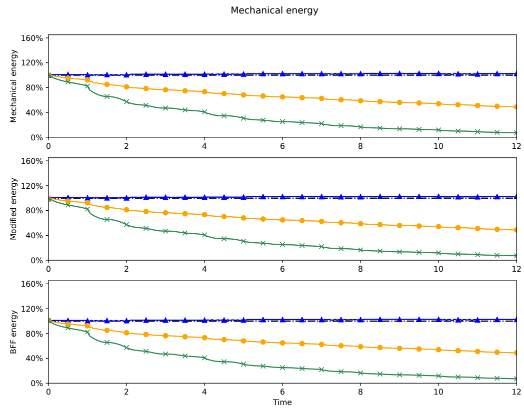
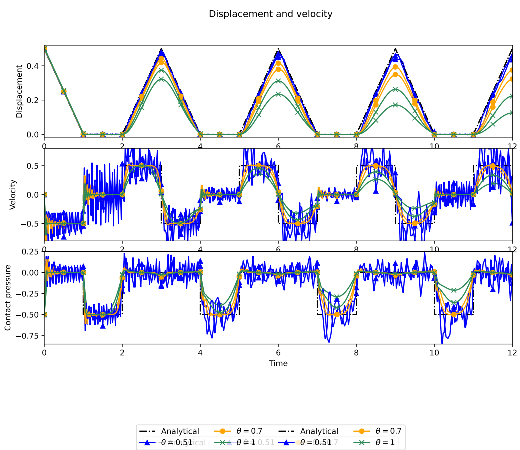


(C)



(D)



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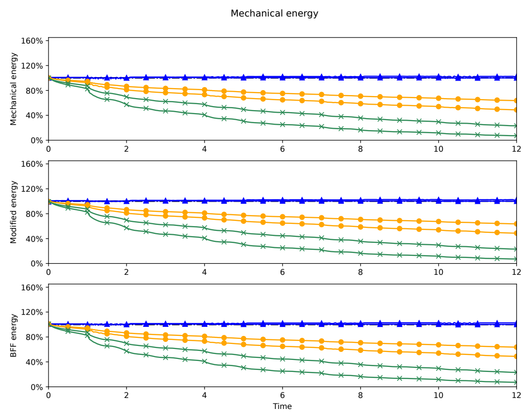


(F)

Figure 5.5. Newmark scheme with (top) $h = \Delta t = 0.1$; (middle) $h = \Delta t = 0.05$; (bottom) $h = \Delta t = 0.025$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).

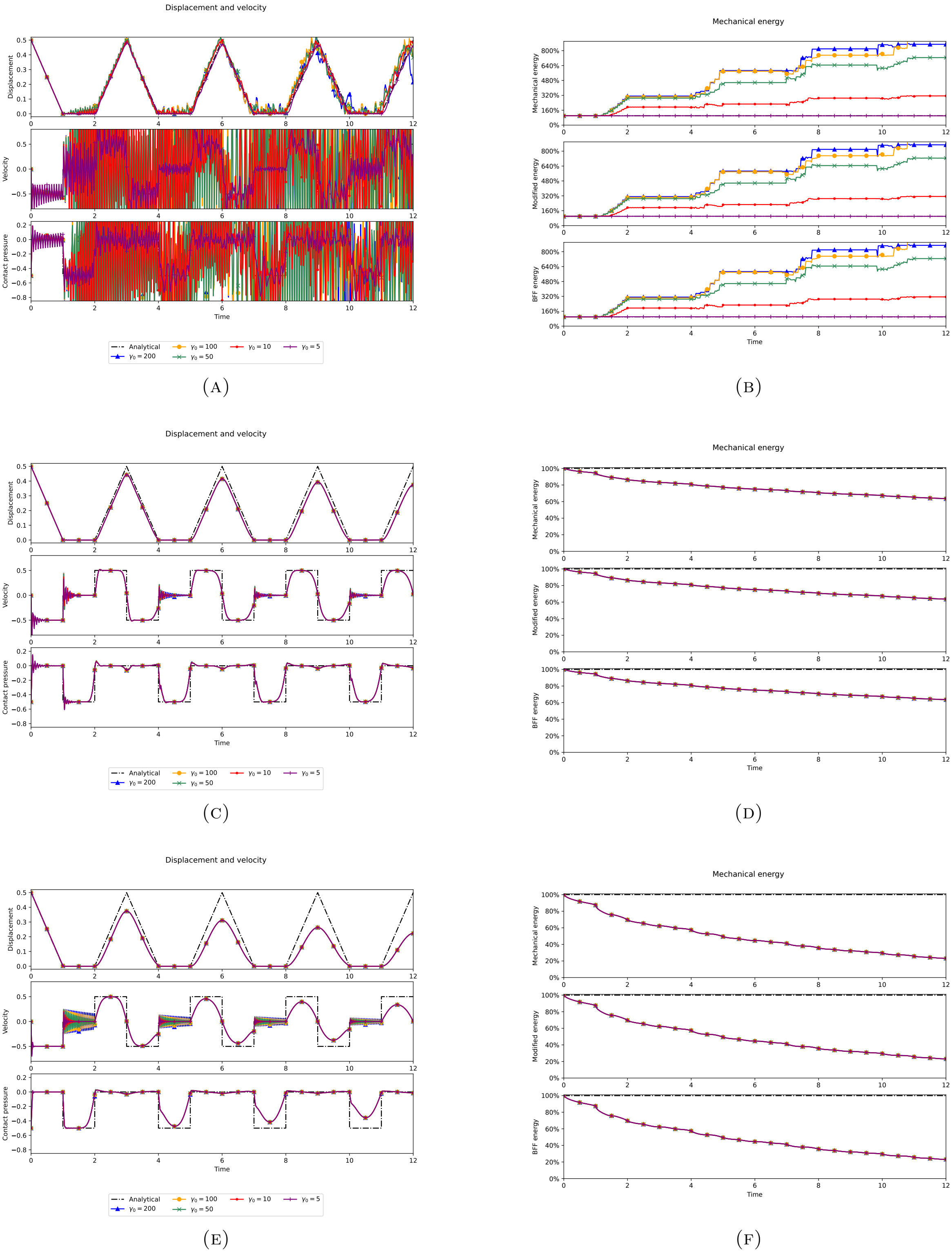
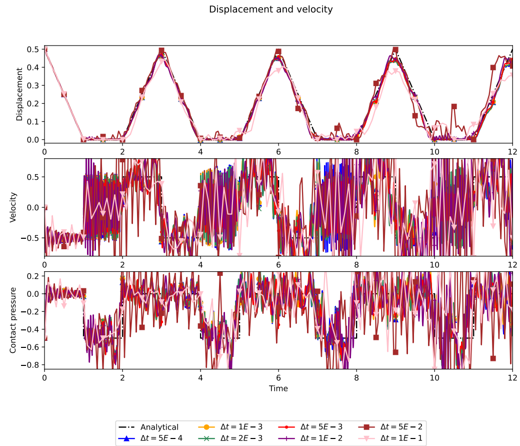
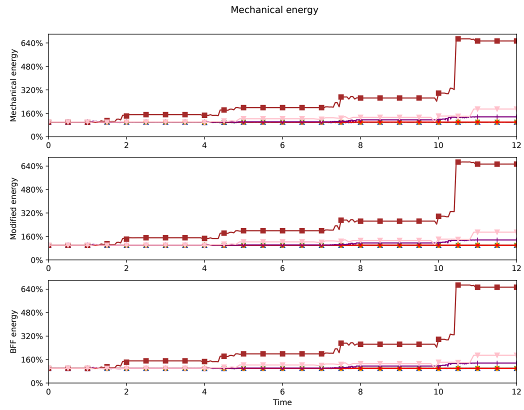


Figure 5.6. Newmark scheme using a variable value of γ_0, with (top) Crank–Nicolson; (middle) $\beta = 0.3$, $\zeta = 0.6$; (bottom) $\beta = \frac{1}{2}$, $\zeta = 1$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).



(A)



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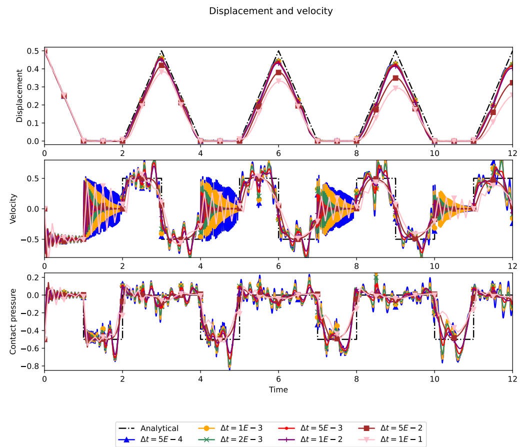

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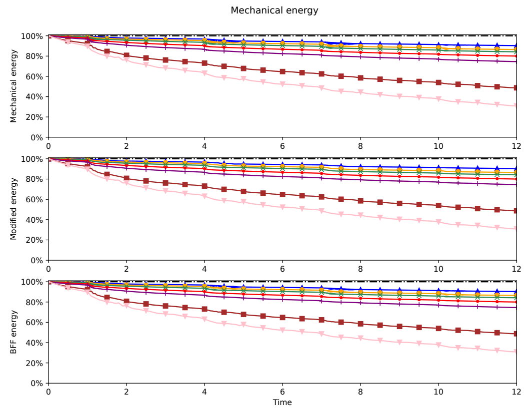

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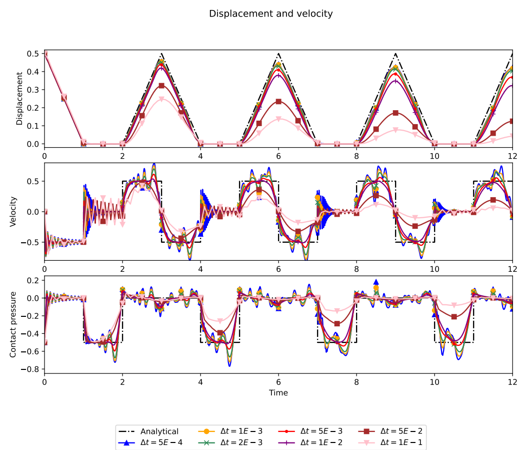

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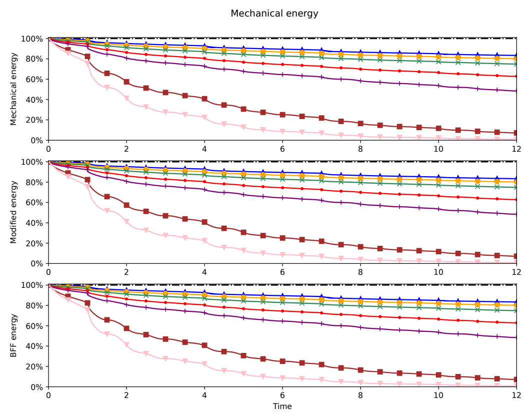
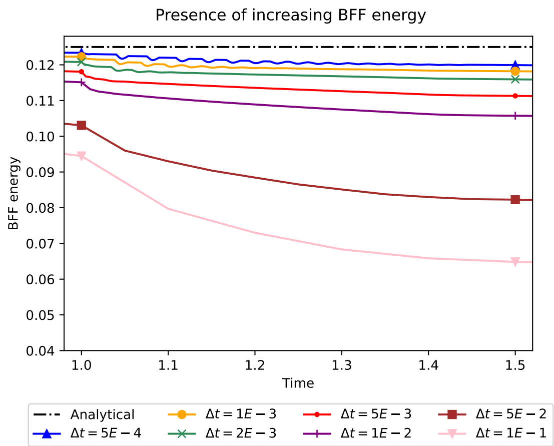
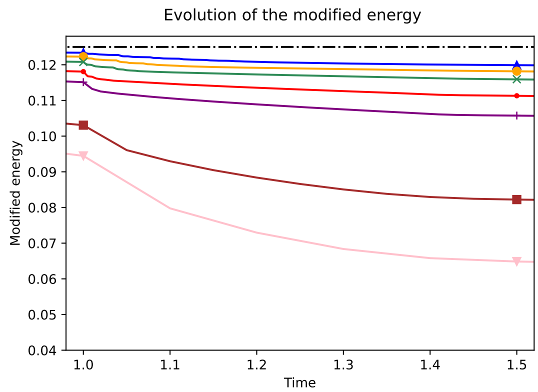


(F)

FIGURE 5.7. Newmark scheme using $h = 0.05$ and a variable Δt, with (top) Crank–Nicolson; (middle) $\beta = 0.3$, $\zeta = 0.6$; (bottom) $\beta = \frac{1}{2}$, $\zeta = 1$. Left: displacement, velocity and contact stress. Right: energies (mechanical, modified, BFF).



(A)



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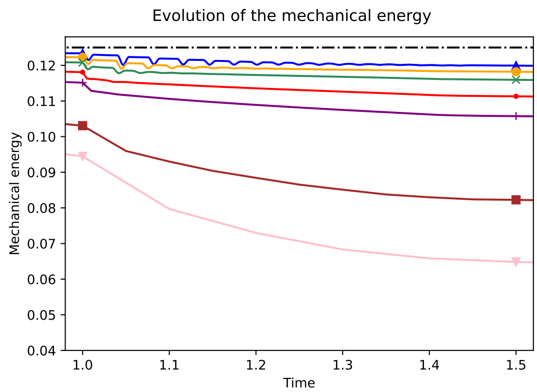

(C)

FIGURE 5.8. The energy evolution at the beginning of the first impact for Newmark scheme with $\beta = \frac{1}{2}, \zeta = 1$. (a) BFF energy; (b) Modified energy; (c) mechanical energy.

6. Conclusion

In this study, we investigated the stability and performance of the θ-scheme and Newmark's scheme using symmetric Nitsche's method. Moreover, we have realized a serie of numerical simulations providing insights into the behavior of these schemes under different parameter values which confirms theoretical results. Our main findings can be summarized as follows:

- The backward Euler scheme (θ-scheme with $\theta = 1$) and Newmark's scheme with $\beta = \frac{1}{2}$ and $\zeta = 1$ demonstrate unconditional stability. This means that the modified energy $E^{h,n}_{\Theta_N}$ is non-increasing, ensuring that these schemes are both dissipative and non-amplifying.
- For both modified energy and BFF energy, the θ-scheme with $\theta \in (\frac{1}{2}, 1)$ and the Newmark scheme with $\zeta = 2\beta$, $\zeta \in (\frac{1}{2}, 1)$ satisfy a property analogous to the discrete Gronwall lemma.
- For highly dissipative schemes, i.e., $\theta \simeq 1$ or $\zeta \simeq 1$, using a larger value of γ_0 increase stability.
- For low dissipative schemes, a moderate value of γ_0 helps to reduce parasitic oscillations and improve stability. However, γ_0 must be large enough to ensure well-posedness.
- Smaller time-steps generally improve numerical accuracy and reduce energy loss without increasing oscillations (this confirms particularly the results of [30] and [33]).

In conclusion, the θ-scheme and Newmark's schemes are effective for stable and accurate simulations of elastodynamic Signorini problems discretized with finite elements and Nitsche's method with carefully chosen parameters.

Acknowledgements

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