SMAI-JCM
SMAI Journal of Computational Mathematics

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Volume 4 (2018), p. 151-195.
<http://smai-jcm.cedram.org/item?id=SMAI-JCM_2018__4__151_0>

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Gradient flow dynamics of two-phase biomembranes: Sharp interface variational formulation and finite element approximation

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Abstract. A finite element method for the evolution of a two-phase membrane in a sharp interface formulation is introduced. The evolution equations are given as an L^2–gradient flow of an energy involving an elastic bending energy and a line energy. In the two phases Helfrich-type evolution equations are prescribed, and on the interface, an evolving curve on an evolving surface, highly nonlinear boundary conditions have to hold. Here we consider both C^0– and C^1–matching conditions for the surface at the interface. A new weak formulation is introduced, allowing for a stable semidiscrete parametric finite element approximation of the governing equations. In addition, we show existence and uniqueness for a fully discrete version of the scheme. Numerical simulations demonstrate that the approach can deal with a multitude of geometries. In particular, the paper shows the first computations based on a sharp interface description, which are not restricted to the axisymmetric case.

2010 Mathematics Subject Classification. 35R01, 49Q10, 65M12, 65M60, 82B26, 92C10.

Keywords. parametric finite elements, Helfrich energy, spontaneous curvature, multi-phase membrane, line energy, C^0– and C^1–matching conditions.

1. Introduction

Two-phase elastic membranes, consisting of coexisting fluid domains, have received a lot of attention in the last 20 years. The interest in two-phase membranes in particular was triggered by the multitude of different shapes observed in experiments with inhomogeneous biomembranes and vesicles. Biomembranes are typically formed as a lipid bilayer, and often multiple lipid components are involved, which laterally can separate into coexisting phases with different properties. Among the complex morphologies that appear are micro-domains, which resemble lipid rafts, and these are of huge interest in biology and medicine. As the thickness of the membrane is much smaller than its lateral length scale, typically the membrane is modelled as a two-dimensional hypersurface in three dimensional Euclidean space. The equilibrium shape of the membrane is obtained by minimizing an energy which –besides other contributions– contains bending energies involving the mean curvature and the Gaussian curvature of the membrane. If different phases occur, parameters in the curvature energy are inhomogeneous, leading to an interesting free boundary problem as well as to a plethora of different shapes. We refer to [12], where multi-component giant unilamellar vesicles (GUVs) separating into different phases were studied. These authors were able to optically resolve interactions between the different phases, its curvature elasticity and the line tension of its interface.

There have been several studies on theoretical and numerical aspects of two-phase membranes taking curvature elasticity and line energy into account, see e.g. [28, 29, 39, 11, 40, 15, 30, 22, 23, 24, 25, 26, 13, 32, 14, 9], which we discuss in the following.

The by now classical model for a one-phase membrane rests on the Canham–Helfrich–Evans elastic bending energy

$$\tfrac{1}{2}\,\alpha \int_\Gamma (\varkappa - \overline{\varkappa})^2 \,\mathrm{d}\mathcal{H}^2 + \alpha^G \int_\Gamma \mathcal{K} \,\mathrm{d}\mathcal{H}^2\,,$$

where Γ is a closed two-dimensional hypersurface and $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure. The mean curvature of Γ is denoted by $\varkappa$, and $\mathcal{K}$ is its Gaussian curvature. The constants α and α^G are bending rigidities, while $\overline{\varkappa}$ is the spontaneous curvature reflecting asymmetry in the membrane introduced, for instance, by different environments on both sides of the membrane.

In a fundamental work, Jülicher and Lipowsky ([28, 29]) generalized the Canham–Helfrich–Evans model to two-phase membranes. The geometry is now given by two smooth surfaces Γ_1 and Γ_2, with a common boundary γ. In general, the constants α, α^G and $\overline{\varkappa}$ take different values in the two phases Γ_1 and Γ_2, which we will denote with an index i. On the curve γ line tension effects play an important role, and the total energy introduced in [28, 29] is given as

$$E((\Gamma_i)_{i=1}^2) = \sum_{i=1}^2 \left[\tfrac{1}{2}\,\alpha_i \int_{\Gamma_i} (\varkappa_i - \overline{\varkappa}_i)^2 \,\mathrm{d}\mathcal{H}^2 + \alpha_i^G \int_{\Gamma_i} \mathcal{K}_i \,\mathrm{d}\mathcal{H}^2\right] + \varsigma\,\mathcal{H}^1(\gamma)\,, \tag{1.1}$$

where the constant $\varsigma \in \mathbb{R}_{\geq 0}$ denotes a possible line tension, and where an index $i \in \{1, 2\}$ states that quantities such as the curvatures and physical constants are evaluated with respect to Γ_i. Of course, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

In [29] it is assumed that the surface $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ is a C^1–surface, meaning in particular that the normal to Γ is continuous across the phase boundary γ. The works [25, 26, 27], on the other hand, also allow for discontinuities of the normal at γ. The first variation of the energy E in (1.1) has been derived in [22] for the C^1–case and in [41] for the C^1– and the C^0–case. It is the goal of this paper to develop a numerical method for a gradient flow evolution of the energy E. To be more precise, we will consider an evolution of the form

$$\left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_\Gamma + \varrho \left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_\gamma = \left[\frac{\delta}{\delta\Gamma}\,E((\Gamma_i)_{i=1}^2)\right](\vec{\chi})\,. \tag{1.2}$$

Here $\vec{\mathcal{V}}$ is the velocity of the surface, $\frac{\delta}{\delta\Gamma}\,E$ is the first variation of the energy, $\vec{\chi}$ is a test vector field on the surface related to directions in which one perturbs the given surface Γ, and $\varrho \geq 0$ is a given constant. In addition, $\langle\cdot,\cdot\rangle_\Gamma$ and $\langle\cdot,\cdot\rangle_\gamma$ denote the L^2–inner products on the surface Γ and on the curve γ, respectively. The evolution of the surface is hence given as a steepest descent dynamics with respect to a weighted L^2–inner product that combines contributions from the surface and the curve. It will turn out that the governing equations in the case where the surface is restricted to be C^1 are

$$\vec{\mathcal{V}} = [-\alpha_i\,\Delta_s\,\varkappa_i + \tfrac{1}{2}\,\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2\,\varkappa_i - \alpha_i\,(\varkappa_i - \overline{\varkappa}_i)\,|\nabla_s\,\vec{\nu}_i|^2]\,\vec{\nu}_i \quad \text{on } \Gamma_i\,, \tag{1.3}$$

together with the boundary conditions on $\gamma = \partial\Gamma_i$:

$$\alpha_1\,(\varkappa - \overline{\varkappa}_1) + \alpha_1^G\,\vec{\varkappa}_\gamma\,.\,\vec{\nu} = \alpha_2\,(\varkappa_2 - \overline{\varkappa}_2) + \alpha_2^G\,\vec{\varkappa}_\gamma\,.\,\vec{\nu}\,, \tag{1.4a}$$

$$[\alpha_i\,(\nabla_s\,\varkappa_i)]_1^2\,.\,\vec{\mu} - [\alpha_i^G]_1^2\,\tau_s + \varsigma\,\vec{\varkappa}_\gamma\,.\,\vec{\nu} = \varrho\,\vec{\mathcal{V}}\,.\,\vec{\nu}\,, \tag{1.4b}$$

$$-\tfrac{1}{2}\,[\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2]_1^2 + [\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)\,(\varkappa_i - \vec{\varkappa}_\gamma\,.\,\vec{\nu})]_1^2 + [\alpha_i^G]_1^2\,\tau^2 + \varsigma\,\vec{\varkappa}_\gamma\,.\,\vec{\mu} = \varrho\,\vec{\mathcal{V}}\,.\,\vec{\mu}\,. \tag{1.4c}$$

Equation (1.3), with Δ_s and ∇_s denoting the surface Laplacian and the surface gradient on Γ_i, respectively, is Willmore flow taking spontaneous curvature effects into account. The boundary condition (1.4a), with $\vec{\varkappa}_\gamma$ denoting the curvature vector on $\gamma(t)$, generalizes the equation for the mean curvature in Navier boundary conditions, appearing for example in [18, (6)]. The equations (1.4b,c), with τ being the geodesic torsion of the curve $\gamma(t)$ on $\Gamma(t)$ and with $[a_i]_1^2 = a_2 - a_1$ denoting the jump of a across $\gamma(t)$, appear in the case $\varrho = 0$ in [23, (3.17), (3.18)], where additional terms to fix the surface areas and the enclosed volume appear. In the axisymmetric case, the equations (1.4a–c) reduce to the equations studied in [29]. Similar conditions have been derived in [39], and it has already been discussed in [23, Appendix B] that these authors miss one term. For positive ϱ the equations (1.4b,c) give rise to dynamic boundary conditions taking into account an additional dissipation mechanism at the boundary. A similar condition for semi-free boundary conditions has been analyzed in [1, (1.3)]. For evolutions where the surface areas of Γ_1 and Γ_2, as well as the volume enclosed by Γ, are conserved, additional terms appear in (1.3) and (1.4c), see (2.16) and (2.20c), below. Moreover, in the case that the surface Γ is just continuous, the boundary conditions (1.4a–c) have to be replaced, and we refer to (2.19a,b), below, for the relevant equations.

Numerically mainly the C^1–case has been studied, with the exception of [26], where C^0–surfaces with kinks in the axisymmetric case were studied numerically with the help of a phase field method. In the C^1–case already in [29] several two-phase equilibrium shapes in the axisymmetric case were computed by solving a governing boundary value problem for a system of ordinary differential equations. Based on research on model membranes, see [12], it has now become possible to perform a systematic analysis of the influence of parameters also in the case of two-phase coexistence. We refer to [11], where experimental vesicle shapes were compared with shapes obtained by solving numerically the axisymmetric shape equations derived in [29]. In this context, we also refer to [14], where, in contrast to the above works, also the effect of spontaneous curvature is taken into account in the axisymmetric case. These authors were able to show that spontaneous curvatures already in an axisymmetric setup give rise to a multitude of morphologies not seen in the case without spontaneous curvature.

Almost all numerical results mentioned so far were for a sharp interface setup. Another successful approach uses a phase field to describe the two phases on the membrane. Line energy in this context is replaced by a Ginzburg–Landau energy like in the classical Cahn–Hilliard theory. We refer to [40, 30, 22, 23, 24, 25, 26, 32, 31] for numerical results based on the phase field approach. The above papers use a gradient flow approach to obtain equilibrium shapes in the large time limit. An evolution law using a Cahn–Hilliard equation on the membrane coupled to surface and bulk (Navier–)Stokes equations has been studied by the present authors in [9].

Rigorous analytical results for two-phase elastic membranes are very limited. So far only results for the axisymmetric case are known. We refer to the work [13], where the existence of global minimizers for axisymmetric multi-phase membranes was shown, and the works [25, 26, 27], where the sharp interface limit of the phase field approach in an axisymmetric situation was studied. Existence results for the evolution problem are not available in the literature so far and should be addressed in the future.

It is the goal of this paper to introduce a finite element approximation for a gradient flow dynamics of the membrane energy E, which is based on a sharp interface approach. Instead of using a phase field on the membrane, we will directly discretize the curve γ separating the two phases Γ_1 and Γ_2. In three dimensions the total surface Γ will be discretized with the help of polyhedral surfaces consisting of a union of triangles. The curve γ is discretized as a polygonal curve in $\mathbb{R}^3$ fitted to the discretization of Γ in the sense that the polygonal curve is the boundary of the open polyhedral sets Γ_1 and Γ_2. The boundary conditions (1.4a–c) are highly nonlinear and involve derivatives of an order up to three when formulated with the help of a parameterization. It is hence highly non-trivial to discretize them in a piecewise linear setup. In this work, a splitting method is used, which basically uses the position vectors of the nodes and an approximation of the mean curvature vector as unknowns. The approach in this paper relies on a discretization of mean curvature leading to good mesh properties. This discretization was introduced by the present authors in [2, 3] and has been previously used for closed and open membranes, see [8, 10] and for elastic curvature flow of curves with junctions, see [6].

We will use the variational structure of the problem to derive a discretization which will turn out to be stable in a spatially discrete and continuous-in-time semidiscrete formulation. In order to do so, we will make use of an appropriate Lagrangian and will use ideas of PDE constrained optimization.

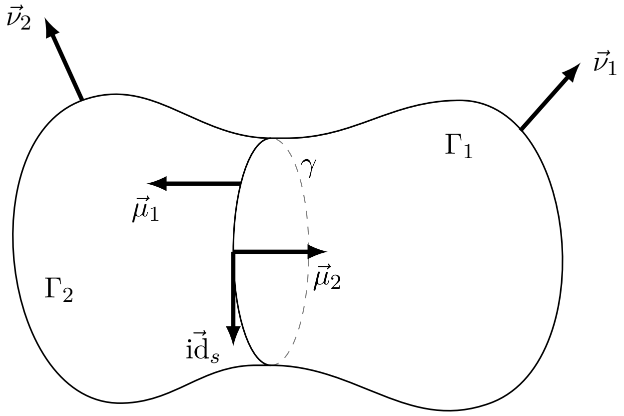

Figure 2.1. Sketch of $\Gamma = \Gamma_1 \cup \gamma \cup \Gamma_2$ with outer unit normals $\vec\nu_i$, conormals $\vec\mu_i$ and tangent vector $\vec{\rm id}_s$ on γ for the case $d = 3$.

The outline of this paper is as follows. In the subsequent section we will formulate the governing equations with all the details. In Section 3 a weak formulation is introduced using the calculus of PDE constrained optimization. A semidiscrete discretization is formulated in Section 4. For this scheme also energy decay properties and conservation properties are shown. In Section 5 a fully discrete version of the scheme is introduced, leading to a linear system at each time level, which is shown to be uniquely solvable. In Section 6 we discuss ideas on how to solve the resulting linear algebra problems numerically. In Section 7 we present several numerical results showing that the new approach allows to approximate solutions to the governing equations also in highly nontrivial geometries. In an appendix we show that the weak formulation derived in this work yields in fact the strong formulation for sufficiently smooth evolutions.

2. The governing equations

In this section we precisely formulate the governing equations both for the C^0– and the C^1–case. We always assume that $(\Gamma(t))_{t\in[0,T]}$ is an evolving hypersurface without boundary in $\mathbb{R}^d$, $d = 2, 3$, that is parameterized by $\vec x(\cdot, t) : \Upsilon \to \mathbb{R}^d$, where $\Upsilon \subset \mathbb{R}^d$ is a given reference manifold, i.e. $\Gamma(t) = \vec x(\Upsilon, t)$. Then

$$\vec{\mathcal{V}}(\vec q, t) := \vec x_t(\vec z, t) \qquad \forall\, \vec q = \vec x(\vec z, t) \in \Gamma(t) \tag{2.1}$$

defines the velocity of $\Gamma(t)$. In order to introduce the two-phase aspect, we consider the decomposition $\Gamma(t) = \Gamma_1(t) \cup \gamma(t) \cup \Gamma_2(t)$, where the interiors of $\Gamma_1(t)$ and $\Gamma_2(t)$ are disjoint and $\gamma(t) = \partial\Gamma_1(t) = \partial\Gamma_2(t)$. We assume that each $\Gamma_i(t)$ is smooth, with outer unit normal $\vec\nu_i(t)$. See Figure 2.1 for a sketch of the setup in the case $d = 3$. In particular, we parameterize the two parts of the surface over fixed oriented, compact, smooth reference manifolds $\Upsilon_i \subset \Upsilon$, i.e. we let $\Gamma_i(t) = \vec x(\Upsilon_i, t)$, $i = 1, 2$. Throughout this paper we will investigate two different types of junction conditions on $\gamma(t)$:

$$C^0\text{–case}: \quad \gamma(t) = \partial\Gamma_1(t) = \partial\Gamma_2(t)\,, \tag{2.2a}$$

$$C^1\text{–case}: \quad \gamma(t) = \partial\Gamma_1(t) = \partial\Gamma_2(t) \;\text{ and }\; \vec\nu_1 = \vec\nu_2 \quad \text{on } \gamma(t)\,. \tag{2.2b}$$

Of course, in the case (2.2b) it also holds that $\vec\mu_1 = -\vec\mu_2$, where $\vec\mu_i$ denotes the outer conormal to $\Gamma_i(t)$ on $\gamma(t)$.

In order to formulate the governing problems in more detail, we denote by $\nabla_s = (\partial_{s_1}, \ldots, \partial_{s_d})$ the surface gradient on Γ_i, and then define $\nabla_s\,\vec\chi = \left(\partial_{s_j}\,\chi_k\right)_{k,j=1}^{d}$, as well as the Laplace–Beltrami operator $\Delta_s = \nabla_s\,.\,\nabla_s = \sum_{j=1}^d \partial_{s_j}^2$. We then introduce the mean curvature vector as

$$\vec\varkappa_i = \varkappa_i\,\vec\nu_i = \Delta_s\,\vec{\rm id} \qquad \text{on } \Gamma_i\,, \tag{2.3}$$

where $\vec{\rm id}$ is the identity function on $\mathbb{R}^d$, and $\varkappa_i$ is the mean curvature of Γ_i, i.e. the sum of the principal curvatures of Γ_i. In particular, the principal curvatures $\varkappa_{i,j}$, $j = 1,\ldots,d-1$, together with the eigenvalue zero for the eigenvector $\vec\nu_i$, are the d eigenvalues of the symmetric linear map $-\nabla_s\,\vec\nu_i : \mathbb{R}^d \to \mathbb{R}^d$; see e.g. [17, p. 152], where a different sign convention is used. The map $-\nabla_s\,\vec\nu_i$ is also called the Weingarten map or shape operator. The mean curvature $\varkappa_i$ and the Gaussian curvature $\mathcal{K}_i$ of Γ_i can now be stated as

$$\varkappa_i = \sum_{j=1}^{d-1} \varkappa_{i,j} = -\operatorname{tr}(\nabla_s\,\vec\nu_i) = -\nabla_s\,.\,\vec\nu_i \qquad \text{and} \qquad \mathcal{K}_i = \prod_{j=1}^{d-1} \varkappa_{i,j}\,. \tag{2.4}$$

Throughout the paper the main case we are interested in is $d = 3$, but it is often convenient to also discuss the case $d = 2$ at the same time. To this end, we generalize the free energy (1.1) to

$$E((\Gamma_i(t))_{i=1}^2) = \sum_{i=1}^2 \left[\tfrac12\,\alpha_i \int_{\Gamma_i(t)} (\varkappa_i - \overline\varkappa_i)^2 \,{\rm d}\mathcal{H}^{d-1} + \alpha_i^G \int_{\Gamma_i(t)} \mathcal{K}_i \,{\rm d}\mathcal{H}^{d-1}\right] + \varsigma\,\mathcal{H}^{d-2}(\gamma(t))\,, \tag{2.5}$$

where $\varkappa_i$ and $\mathcal{K}_i$ are the mean and Gaussian curvatures of $\Gamma_i(t)$, $i = 1, 2$, $\varsigma \in \mathbb{R}_{\geq 0}$ denotes a possible line tension, and $\alpha_i \in \mathbb{R}_{>0}$ and $\alpha_i^G \in \mathbb{R}$ denote the bending and Gaussian bending rigidities of $\Gamma_i(t)$, $i = 1, 2$, respectively. Here and throughout $\mathcal{H}^k$, $k = 0, 1, 2$, denotes the k-dimensional Hausdorff measure in $\mathbb{R}^d$.

In the case $d = 2$, we always assume that $\varsigma = \alpha_1^G = \alpha_2^G = 0$. For the case $d = 3$, on the other hand, we mention that the contributions

$$\sum_{i=1}^2 \left[\tfrac12\,\alpha_i \int_{\Gamma_i(t)} \varkappa_i^2 \,{\rm d}\mathcal{H}^2 + \alpha_i^G \int_{\Gamma_i(t)} \mathcal{K}_i \,{\rm d}\mathcal{H}^2\right] \tag{2.6}$$

to the energy (2.5) are positive semidefinite with respect to the principal curvatures if

$$\alpha_i^G \in [-2\,\alpha_i, 0]\,, \quad i = 1, 2\,. \tag{2.7}$$

In the C^1–case, recall (2.2b), adding multiples of $\sum_{i=1}^2 \mathcal{K}_i \,{\rm d}\mathcal{H}^2$ to the energy only changes the energy by a constant which follows from the Gauss–Bonnet theorem, see (2.12) below. Hence we obtain that the energy (2.5) can be bounded from below if $\alpha_i^G \geq \max\{\alpha_1^G, \alpha_2^G\} - 2\,\alpha_i$ for $i = 1, 2$, which will hold whenever

$$\min\{\alpha_1, \alpha_2\} \geq \tfrac12\,|\alpha_1^G - \alpha_2^G|\,. \tag{2.8}$$

Variational problems for integrals including the energy (2.6) require that the energy is definite, see e.g. [33, p. 364], in order to be able to show a priori estimates. As discussed in [33], the condition of definiteness leads to the constraints (2.7) and (2.8), and it is likely that these conditions also have implications for the existence and regularity theory of gradient flows for (2.5) in the C^0– and C^1–case, respectively.

In the case $d = 3$, similarly to (2.3), fundamental to many approaches, which numerically approximate evolving curves in a parametric way, is the identity

$$\vec{\rm id}_{ss} = \vec\varkappa_\gamma \quad \text{on } \gamma(t)\,, \tag{2.9}$$

where $\vec\varkappa_\gamma$ is the curvature vector on $\gamma(t)$. Here we choose the arclength s of the curve $\gamma(t)$ such that

$$\vec\mu_i = (-1)^i\,\vec\nu_i \times \vec{\rm id}_s \quad \text{on } \gamma(t), \tag{2.10}$$

for $i = 1, 2$, denote the outer conormals to $\Gamma_i(t)$ on $\gamma(t)$. Note that $\vec\mu_i$ is a vector that is perpendicular to the unit tangent $\vec{\mathrm{id}}_s$ on $\partial\Gamma_i(t)$ and lies in the tangent space of $\Gamma_i(t)$. Now (2.9) can be rewritten as

$$\vec{\mathrm{id}}_{ss} = \vec\varkappa_\gamma = (\vec\varkappa_\gamma \,.\, \vec\mu_i)\, \vec\mu_i + (\vec\varkappa_\gamma \,.\, \vec\nu_i)\, \vec\nu_i \quad \text{on} \ \ \gamma(t)\,, \tag{2.11}$$

where $\vec\varkappa_\gamma \,.\, \vec\mu_i$ is the geodesic curvature and $\vec\varkappa_\gamma \,.\, \vec\nu_i$ is the normal curvature of $\gamma(t)$ on $\Gamma_i(t)$, $i = 1, 2$. It then follows from the Gauss–Bonnet theorem,

$$\int_{\Gamma_i(t)} \mathcal{K}_i \,\mathrm{d}\mathcal{H}^2 = 2\,\pi\, m(\Gamma_i(t)) + \int_{\gamma(t)} \vec\varkappa_\gamma \,.\, \vec\mu_i \,\mathrm{d}\mathcal{H}^1, \tag{2.12}$$

where $m(\Gamma_i(t)) \in \mathbb{Z}$ denotes the Euler characteristic of $\Gamma_i(t)$, that the energy (2.5), is equivalent to

$$E((\Gamma_i(t))_{i=1}^2) = \sum_{i=1}^2 \left[\tfrac12\,\alpha_i \int_{\Gamma_i(t)} (\varkappa_i - \overline\varkappa_i)^2 \,\mathrm{d}\mathcal{H}^2 + \alpha_i^G \left[\int_{\gamma(t)} \vec\varkappa_\gamma \,.\, \vec\mu_i \,\mathrm{d}\mathcal{H}^1 + 2\,\pi\, m(\Gamma_i(t)) \right] \right] + \varsigma\, \mathcal{H}^1(\gamma(t))\,. \tag{2.13}$$

We note that we use a sign for the conormal that is different from many authors in differential geometry, and hence we obtain a different sign in the Gauss–Bonnet formula.

In some cases, in particular in applications for biomembranes, cf. [38], the surface areas of $\Gamma_1(t)$ and $\Gamma_2(t)$ need to stay constant during the evolution, as well as the volume enclosed by $\Gamma(t)$. Here and throughout we use the terminology "surface area" and "enclosed volume" also for the case $d = 2$, when the former is really curve length, and the latter means enclosed area. In this case one can consider

$$E_\lambda((\Gamma_i(t))_{i=1}^2) = E((\Gamma_i(t))_{i=1}^2) + \lambda^V(t)\, \mathcal{L}^d(\Omega(t)) + \sum_{i=1}^2 \lambda_i^A(t)\, \mathcal{H}^{d-1}(\Gamma_i(t))\,, \tag{2.14}$$

where $\Omega(t)$ denotes the interior of $\Gamma(t)$ and $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$. Here, $\lambda_i^A(t)$ are Lagrange multipliers for the area constraints, which can be interpreted as a surface tension, and $\lambda^V(t)$ is a Lagrange multiplier for the volume constraint which might be interpreted as a pressure difference.

For the convenience of the reader, we end this section by stating the strong formulations of the L^2–gradient flows for (2.5) in the presence of the matching conditions (2.2a) and (2.2b), respectively. These strong formulations directly follow from the weak formulation introduced in Section 3, as we show rigorously in the appendix.

The weighted L^2–gradient flow, (1.2), of (2.13), for $d = 2$ or $d = 3$, then leads to the evolution law

$$\vec{\mathcal{V}} \,.\, \vec\nu_i = -\alpha_i\, \Delta_s\, \varkappa_i + \tfrac12\, \alpha_i\, (\varkappa_i - \overline\varkappa_i)^2\, \varkappa_i - \alpha_i\, (\varkappa_i - \overline\varkappa_i)\, |\nabla_s\, \vec\nu_i|^2 \quad \text{on}\ \Gamma_i(t)\,,\ \ i = 1, 2\,. \tag{2.15}$$

See (A.8) in the appendix for a derivation of (2.15). We remark that if the more general energy (2.14) is considered, then (2.15) is replaced by

$$\vec{\mathcal{V}} \,.\, \vec\nu_i = -\alpha_i\, \Delta_s\, \varkappa_i + \tfrac12\, \alpha_i\, (\varkappa_i - \overline\varkappa_i)^2\, \varkappa_i - \alpha_i\, (\varkappa_i - \overline\varkappa_i)\, |\nabla_s\, \vec\nu_i|^2 + \lambda_i^A\, \varkappa_i - \lambda^V \quad \text{on}\ \Gamma_i(t)\,, \tag{2.16}$$

for $i = 1, 2$, see (A.13) in the appendix.

In the case $d = 3$ we introduce the second fundamental form $\mathrm{I\!I}_i$ of $\Gamma_i(t)$, which is given as

$$\mathrm{I\!I}_i(\vec{\mathfrak{t}}_1, \vec{\mathfrak{t}}_2) = -[\partial_{\vec{\mathfrak{t}}_1}\, \vec\nu_i] \,.\, \vec{\mathfrak{t}}_2 = -[(\nabla_s\, \vec\nu_i)\, \vec{\mathfrak{t}}_1] \,.\, \vec{\mathfrak{t}}_2 \qquad \text{on}\ \ \Gamma_i(t)\,, \tag{2.17}$$

for all tangential vectors $\vec{\mathfrak{t}}_j$, $j = 1, 2$. We note that $\mathrm{I\!I}_i(\cdot,\cdot)$ is a symmetric bilinear form, as $\nabla_s\, \vec\nu_i$ is symmetric. In addition, we define

$$\tau_i = \mathrm{I\!I}_i(\vec{\mathrm{id}}_s, \vec\mu_i) \qquad \text{on}\ \gamma(t)\,, \tag{2.18}$$

i.e. $\tau_i = -(\vec\nu_i)_s \,.\, \vec\mu_i$ on $\gamma(t)$.

Still considering the case $d = 3$, in the C^0–junction case, the boundary conditions on $\gamma(t)$ are given by

$$\alpha_i\, (\varkappa_i - \overline\varkappa_i) + \alpha_i^G\, \vec\varkappa_\gamma \,.\, \vec\nu_i = 0 \quad \text{on}\ \ \gamma(t)\,, \quad i = 1, 2\,, \tag{2.19a}$$

$$\sum_{i=1}^{2}\Big[((\alpha_i\,(\nabla_s\,\varkappa_i)\,.\,\vec\mu_i-\alpha_i^G\,(\tau_i)_s)\,\vec\nu_i-(\tfrac12\,\alpha_i\,(\varkappa_i-\overline{\varkappa}_i)^2+\alpha_i^G\,\mathcal{K}_i+\lambda_i^A)\,\vec\mu_i\Big]+\varsigma\,\vec\varkappa_\gamma=\varrho\,\vec{\mathcal{V}}\quad\text{on}\ \ \gamma(t)\,, \tag{2.19b}$$

see (A.12a), (A.14) in the appendix. We note that (2.19a) are two scalar conditions, while (2.19b) gives rise to two conditions as $\vec\mu_i$, $\vec\nu_i$ and $\vec\varkappa_\gamma$ are all perpendicular to the tangent space to $\gamma(t)$. Expressing Γ_1 and Γ_2 locally as two graphs, we also obtain one condition for the height functions stemming from the C^0–condition. Altogether we have five conditions, as is to be expected for a free boundary problem involving fourth order operators on both sides of the free boundary. In this context we also refer to Remark 2.1 in [6].

In the C^1–junction case, when $\vec\nu=\vec\nu_1=\vec\nu_2$ and $\vec\mu=\vec\mu_2=-\vec\mu_1$ on $\gamma(t)$, the boundary conditions on $\gamma(t)$ for the dissipation dynamics (1.2), with E replaced by E_λ, are given by

$$[\alpha_i\,(\varkappa_i-\overline{\varkappa}_i)]_1^2+[\alpha_i^G]_1^2\,\vec\varkappa_\gamma\,.\,\vec\nu=0\quad\text{on}\ \ \gamma(t)\,, \tag{2.20a}$$

$$[\alpha_i\,(\nabla_s\,\varkappa_i)]_1^2\,.\,\vec\mu+\varsigma\,\vec\varkappa_\gamma\,.\,\vec\nu-[\alpha_i^G]_1^2\,\tau_s=\varrho\,\vec{\mathcal{V}}\,.\,\vec\nu\quad\text{on}\ \ \gamma(t)\,, \tag{2.20b}$$

$$[-\tfrac12\,\alpha_i\,(\varkappa_i-\overline{\varkappa}_i)^2+\alpha_i\,(\varkappa_i-\overline{\varkappa}_i)\,(\varkappa_i-\vec\varkappa_\gamma\,.\,\vec\nu)-\lambda_i^A]_1^2+[\alpha_i^G]_1^2\,\tau^2+\varsigma\,\vec\varkappa_\gamma\,.\,\vec\mu=\varrho\,\vec{\mathcal{V}}\,.\,\vec\mu\quad\text{on}\ \ \gamma(t)\,, \tag{2.20c}$$

where $\tau=\tau_2=-\tau_1$ is the geodesic torsion of the curve $\gamma(t)$ on $\Gamma(t)$. We note that (2.20a–c), in the case $\varrho=0$, agree with (3.16)–(3.18) in [23], see also [24, (2.7b,a,c)]. In terms of counting the number of equations, we see that (2.20a–c) are three conditions, together with one condition coming from $\vec\nu_1=\vec\nu_2$ and one condition from the requirement that the two phases match up continuously, leading to five conditions in total. We refer to (A.15a), (A.24a,b) in the appendix for a derivation of (2.20a–c).

Remark 2.1. We note that although the conditions (2.19a,b) and (2.20a–c) were derived for the case $d=3$, they are also valid in the case $d=2$ on recalling that in this case we set $\varsigma=\alpha_1^G=\alpha_2^G=0$. In particular, (2.19a) then simplifies to $\varkappa_i=\overline{\varkappa}_i$ on $\gamma(t)$, $i=1,2$, which is the same as the condition [6, (2.13c)] that was derived by the authors for a C^0–junction between two curves meeting in 2d. In addition, (2.19b) for $d=2$ and $\varrho=0$ collapses to [6, (2.13b)], modulo the different sign convention employed there.

Similarly, (2.20a) for $d=2$ simplifies to $\alpha_1\,(\varkappa_1-\overline{\varkappa}_1)=\alpha_2\,(\varkappa_2-\overline{\varkappa}_2)$ on $\gamma(t)$, which is the same as the condition [6, (2.18e)], modulo the different sign convention employed there, that was derived by the authors for a C^1–junction between two curves meeting in 2d. In addition, (2.20b,c) for $d=2$ and $\varrho=0$, collapse to [6, (2.18b,c)].

3. Weak formulation

In this section we derive a weak formulation of a generalized L^2–gradient flow of $E((\Gamma_i(t))_{i=1}^2)$. The weak formulation of the standard L^2–gradient flow is given by (3.29), below, with $\varrho=0$, where f_Γ represents the first variation of the energy $E((\Gamma_i(t))_{i=1}^2)$ formulated in a suitable weak form. In what follows we will define a Lagrangian involving the energy and suitable constraints, which for example relate the curvatures to the parametrizations of the surfaces. This Lagrangian will allow us to derive (3.28a–f), below, which defines f_Γ in a weak formulation involving only first order derivatives. This formulation will be suitable for a numerical approximation based on continuous, piecewise linear finite elements, and such an approximation will be considered in Section 4.

On recalling (2.1), we define the following time derivative that follows the parameterization $\vec x(\cdot,t)$ of $\Gamma(t)$. Let

$$(\partial_t^\circ\,\phi)\,|_{\Gamma_i(t)}=(\phi_t+\vec{\mathcal{V}}\,.\,\nabla\,\phi)\,|_{\Gamma_i(t)}\qquad\forall\ \phi\in H^1(\Gamma_{i,T})\,, \tag{3.1}$$

where we have defined the space-time surfaces

$$\Gamma_{i,T} := \bigcup_{t\in[0,T]} \Gamma_i(t) \times \{t\}\,,\ i = 1,2\,, \quad \text{and} \quad \Gamma_T := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}\,.$$

Here we stress that (3.1) is well-defined, even though ϕ_t and $\nabla\,\phi$ do not make sense separately for a function $\phi \in H^1(\Gamma_{i,T})$. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle \psi_i, \phi_i\rangle_{\Gamma_i(t)} = \langle \partial_t^\circ\,\psi_i, \phi\rangle_{\Gamma_i(t)} + \langle \psi_i, \partial_t^\circ\,\phi_i\rangle_{\Gamma_i(t)} + \left\langle \psi_i\,\phi_i, \nabla_s\,.\,\vec{\mathcal{V}}\right\rangle_{\Gamma_i(t)} \qquad \forall\ \psi_i, \phi_i \in H^1(\Gamma_{i,T})\,, \quad (3.2)$$

see Lemma 5.2 in [21]. Here $\langle\cdot,\cdot\rangle_{\Gamma_i(t)}$ denotes the L^2–inner product on $\Gamma_i(t)$, and $\langle\cdot,\cdot\rangle_{\Gamma(t)} = \sum_{i=1}^2 \langle\cdot,\cdot\rangle_{\Gamma_i(t)}$. It immediately follows from (3.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{H}^{d-1}(\Gamma_i(t)) = \left\langle \nabla_s\,.\,\vec{\mathcal{V}}, 1\right\rangle_{\Gamma_i(t)} = \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}\right\rangle_{\Gamma_i(t)}\,. \quad (3.3)$$

Moreover, on recalling Lemma 2.1 from [17], it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{L}^d(\Omega(t)) = \sum_{i=1}^2 \left\langle \vec{\mathcal{V}}, \vec{\nu}_i\right\rangle_{\Gamma_i(t)}\,. \quad (3.4)$$

In this section we would like to derive a weak formulation for the L^2–gradient flow of $E((\Gamma_i(t))_{i=1}^2)$. To this end, we need to consider variations of the energy with respect to $\Gamma(t) = \vec{x}(\Upsilon, t)$. Let

$$H^1_\gamma(\Gamma(t)) := \{\eta \in L^2(\Gamma(t)) : \eta\,|_{\Gamma_i(t)} \in H^1(\Gamma_i(t)), i = 1,2\,,$$
$$(\eta\,|_{\Gamma_1(t)})\,|_{\gamma(t)} = (\eta\,|_{\Gamma_2(t)})\,|_{\gamma(t)} =: \eta\,|_{\gamma(t)} \in H^1(\gamma(t))\}\,.$$

In addition, for any given $\vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d$ and for any $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 \in \mathbb{R}_{>0}$, let

$$\Gamma_\varepsilon(t) := \{\vec{\Psi}(\vec{z},\varepsilon) : \vec{z} \in \Gamma(t)\}\,, \quad \text{where} \ \ \vec{\Psi}(\vec{z}, 0) = \vec{z} \text{ and } \tfrac{\partial\vec{\Psi}}{\partial\varepsilon}(\vec{z}, 0) = \vec{\chi}(\vec{z}) \quad \forall\ \vec{z} \in \Gamma(t)\,. \quad (3.5)$$

Of course, we have that $\Gamma_\varepsilon(t) = \Gamma_{1,\varepsilon}(t) \cup \gamma_\varepsilon(t) \cup \Gamma_{2,\varepsilon}(t)$, where

$$\Gamma_{i,\varepsilon}(t) := \{\vec{\Psi}(\vec{z},\varepsilon) : \vec{z} \in \Gamma_i(t)\}\,,\ i = 1,2\,, \quad \text{and} \quad \gamma_\varepsilon(t) = \partial\Gamma_{1,\varepsilon}(t) = \partial\Gamma_{2,\varepsilon}(t)\,.$$

Similarly to (3.3), the first variation of $\mathcal{H}^{d-1}(\Gamma_i(t))$ with respect to $\Gamma(t)$ in the direction $\vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d$ is given by

$$\left[\frac{\delta}{\delta\Gamma}\,\mathcal{H}^{d-1}(\Gamma_i(t))\right](\vec{\chi}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\mathcal{H}^{d-1}(\Gamma_{i,\varepsilon}(t))\,|_{\varepsilon=0}$$
$$= \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\left[\mathcal{H}^{d-1}(\Gamma_{i,\varepsilon}(t)) - \mathcal{H}^{d-1}(\Gamma_i(t))\right] = \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi}\right\rangle_{\Gamma_i(t)}\,, \quad (3.6)$$

see e.g. the proof of Lemma 1 in [20].

In order to derive a suitable weak formulation, we formally consider the first variation of (2.5) subject to the following side constraint, which is inspired by the weak formulation of (2.3),

$$\left\langle \underline{\underline{Q}}_{i,\theta}\,\vec{\varkappa}_i^\star, \vec{\eta}\right\rangle_{\Gamma_i(t)} + \left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\eta}\right\rangle_{\Gamma_i(t)} = \langle \vec{\mathrm{m}}_i, \vec{\eta}\rangle_{\gamma(t)} \qquad \forall\ \vec{\eta} \in [H^1(\Gamma_i(t))]^d\,,\ i = 1,2\,, \quad (3.7)$$

where $\theta \in [0,1]$ is a fixed parameter, and where $\underline{\underline{Q}}_{i,\theta}$ are defined by

$$\underline{\underline{Q}}_{i,\theta} = \theta\,\underline{\underline{\mathrm{Id}}} + (1-\theta)\,\vec{\nu}_i \otimes \vec{\nu}_i \quad \text{on}\ \ \Gamma_i(t)\,. \quad (3.8)$$

Of course, (3.7) holds trivially on the continuous level for $\vec{\varkappa}_i^\star = \vec{\varkappa}_i$ and for $\vec{\mathrm{m}}_i$ being the conormal $\vec{\mu}_i$, independently of the choice of $\theta \in [0,1]$. Here we remark that the natural weak formulation of (2.3) would correspond to (3.7) with $\theta = 1$. However, under discretization that formulation would lead to undesirable mesh effects. Hence, in line with the authors previous work in [10], we also allow $\theta \in [0,1)$, which under discretization leads to an induced tangential motion and good meshes for $\theta = 0$, in general. In rare cases we may need to dampen the tangential motion that occurs in the case $\theta = 0$. To this end, we allow for the full range of values $\theta \in [0, 1]$.

Similarly to (3.7), we introduce the following side constraint, inspired by the weak formulation of (2.9):

$$\left\langle \vec{\varkappa}_\gamma^\star, \vec{\eta} \right\rangle_{\gamma(t)} + \left\langle \vec{\mathrm{id}}_s, \vec{\eta}_s \right\rangle_{\gamma(t)} = 0 \quad \forall\, \vec{\eta} \in [H^1(\gamma(t))]^d \,. \tag{3.9}$$

Finally, in order to model a C^0– or C^1–contact we require

$$C_1 \left(\vec{\mathrm{m}}_1 + \vec{\mathrm{m}}_2\right) = \vec{0} \qquad \text{on} \quad \gamma(t) \,, \tag{3.10}$$

where $C_1 = 0$ for C^0 and $C_1 = 1$ for C^1.

We now define the Lagrangian

$$\begin{aligned}
L((\Gamma_i(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) &= \sum_{i=1}^{2} \left[\tfrac{1}{2}\alpha_i \left\langle \vec{\varkappa}_i^\star - \overline{\varkappa}_i\, \vec{\nu}_i, \vec{\varkappa}_i^\star - \overline{\varkappa}_i\, \vec{\nu}_i \right\rangle_{\Gamma_i(t)} + \alpha_i^G \left\langle \vec{\varkappa}_\gamma^\star, \vec{\mathrm{m}}_i \right\rangle_{\gamma(t)} \right] \\
&\quad + \varsigma\, \mathcal{H}^{d-2}(\gamma(t)) - \left\langle \vec{\varkappa}_\gamma^\star, \vec{z} \right\rangle_{\gamma(t)} - \left\langle \vec{\mathrm{id}}_s, \vec{z}_s \right\rangle_{\gamma(t)} + C_1 \left\langle \vec{\mathrm{m}}_1 + \vec{\mathrm{m}}_2, \vec{\phi} \right\rangle_{\gamma(t)} \\
&\quad - \sum_{i=1}^{2} \left[\left\langle \underline{\underline{Q}}_{i,\theta}\, \vec{\varkappa}_i^\star, \vec{y}_i \right\rangle_{\Gamma_i(t)} + \left\langle \nabla_s\, \vec{\mathrm{id}}, \nabla_s\, \vec{y}_i \right\rangle_{\Gamma_i(t)} - \left\langle \vec{\mathrm{m}}_i, \vec{y}_i \right\rangle_{\gamma(t)} \right],
\end{aligned}$$

where $\vec{y}_i \in [H^1(\Gamma_i(t))]^d$ and $\vec{z} \in [H^1(\gamma(t))]^d$ are Lagrange multipliers for (3.7) and (3.9), respectively. Similarly, $\vec{\phi} \in [L^2(\gamma(t))]^d$ is a Lagrange multiplier for (3.10). We now want to compute the first variation f_Γ of $E((\Gamma_i(t))_{i=1}^2)$, subject to the side constraints (3.7), (3.9) and (3.10). This means that f_Γ needs to fulfill

$$f_\Gamma(\vec{\chi}) = -\left[\frac{\delta}{\delta \Gamma}\, E(t)\right](\vec{\chi}) \qquad \forall\, \vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d \,. \tag{3.11}$$

In particular, on using ideas from the formal calculus of PDE constrained optimization, see e.g. [37], we can formally compute f_Γ by requiring that

$$\begin{aligned}
\left[\frac{\delta}{\delta \Gamma}\, L\right](\vec{\chi}) &= \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \Big[(L(\Gamma_{i,\varepsilon}(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \\
&\qquad - L((\Gamma_i(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \Big] = -f_\Gamma(\vec{\chi})
\end{aligned} \tag{3.12a}$$

$$\begin{aligned}
\left[\frac{\delta}{\delta \vec{\varkappa}_1^\star}\, L\right](\vec{\xi}_1) &= \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \Big[L(\Gamma_1(t), \vec{\varkappa}_1^\star + \varepsilon\, \vec{\xi}_1, \vec{\mathrm{m}}_1, \vec{y}_1, \Gamma_2(t), \vec{\varkappa}_2^\star, \vec{\mathrm{m}}_2, \vec{y}_2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \\
&\qquad - L((\Gamma_i(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \Big] = 0 \,,
\end{aligned} \tag{3.12b}$$

$$\begin{aligned}
\left[\frac{\delta}{\delta \vec{\mathrm{m}}_1}\, L\right](\vec{\zeta}_1) &= \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \Big[L(\Gamma_1(t), \vec{\varkappa}_1^\star, \vec{\mathrm{m}}_1 + \varepsilon\, \vec{\zeta}_1, \vec{y}_1, \Gamma_2(t), \vec{\varkappa}_2^\star, \vec{\mathrm{m}}_2, \vec{y}_2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \\
&\qquad - L((\Gamma_i(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \Big] = 0 \,,
\end{aligned} \tag{3.12c}$$

$$\begin{aligned}
\left[\frac{\delta}{\delta \vec{y}_1}\, L\right](\vec{\eta}_1) &= \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \Big[L(\Gamma_1(t), \vec{\varkappa}_1^\star, \vec{\mathrm{m}}_1, \vec{y}_1 + \varepsilon\, \vec{\eta}_1, \Gamma_2(t), \vec{\varkappa}_2^\star, \vec{\mathrm{m}}_2, \vec{y}_2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \\
&\qquad - L((\Gamma_i(t), \vec{\varkappa}_i^\star, \vec{\mathrm{m}}_i, \vec{y}_i)_{i=1}^2, \vec{\varkappa}_\gamma^\star, \vec{z}, \vec{\phi}) \Big] = 0 \,,
\end{aligned} \tag{3.12d}$$

for variations $\vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d$, $\vec{\xi}_1 \in [L^2(\Gamma_1(t))]^d$, $\vec{\zeta}_1 \in [L^2(\gamma(t))]^d$ and $\vec{\eta}_1 \in [L^2(\Gamma_1(t))]^d$; and similarly for the variations for $\vec{\varkappa}_2^\star$, $\vec{\mathrm{m}}_2$, $\vec{y}_2$, $\vec{\varkappa}_\gamma^\star$, $\vec{z}$ and $\vec{\phi}$.

In order to calculate (3.12a–d), we note that generalized variants of (3.6) also hold. Namely, we have that

$$\left[\frac{\delta}{\delta\Gamma}\,\langle w_i, 1\rangle_{\Gamma_i(t)}\right](\vec\chi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\langle w_{i,\varepsilon}, 1\rangle_{\Gamma_{i,\varepsilon}(t)}\mid_{\varepsilon=0} = \left\langle w_i\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i(t)} \quad \forall\ w_i \in L^\infty(\Gamma_i(t))\,, \tag{3.13}$$

where $w_{i,\varepsilon} \in L^\infty(\Gamma_{i,\varepsilon}(t))$, for any $w_i \in L^\infty(\Gamma_i(t))$, is defined by

$$w_{i,\varepsilon}(\vec\Psi(\vec z,\varepsilon)) = w_i(\vec z) \qquad \forall\ \vec z \in \Gamma_i(t)\,,$$

and similarly for $\vec w \in [L^\infty(\Gamma_i(t))]^d$. This definition of $w_{i,\varepsilon}$ yields that $\partial_\varepsilon^0\, w_i = 0$, where

$$\partial_\varepsilon^0\, w_i(\vec z) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} w_{i,\varepsilon}(\vec\Psi(\vec z,\varepsilon))\mid_{\varepsilon=0} \qquad \forall\ \vec z \in \Gamma_i(t). \tag{3.14}$$

Of course, (3.13) is the first variation analogue of (3.2) with $w_i = \psi_i\,\phi_i$ and $\partial_\varepsilon^0\,\psi_i = \partial_\varepsilon^0\,\phi_i = 0$. Similarly, it holds that

$$\begin{aligned}\left[\frac{\delta}{\delta\Gamma}\,\langle \vec w_i, \vec\nu_i\rangle_{\Gamma_i(t)}\right](\vec\chi) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\langle \vec w_{i,\varepsilon}, \vec\nu_{i,\varepsilon}\rangle_{\Gamma_{i,\varepsilon}(t)}\mid_{\varepsilon=0} \\ &= \left\langle (\vec w_i\,.\,\vec\nu_i)\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i(t)} + \left\langle \vec w_i, \partial_\varepsilon^0\,\vec\nu_i\right\rangle_{\Gamma_i(t)} \quad \forall\ \vec w_i \in [L^\infty(\Gamma_i(t))]^d\,,\end{aligned} \tag{3.15}$$

where $\partial_\varepsilon^0\,\vec w_i = \vec 0$ and $\vec\nu_{i,\varepsilon}(t)$ denotes the unit normal on $\Gamma_{i,\varepsilon}(t)$. Moreover, we will make use of the following result concerning the variation of $\vec\nu_i$, with respect to $\Gamma(t)$, in the direction $\vec\chi \in [H^1_\gamma(\Gamma(t))]^d$:

$$\partial_\varepsilon^0\,\vec\nu_i = -[\nabla_s\,\vec\chi]^T\,\vec\nu_i \quad \text{on} \quad \Gamma_i(t) \quad \Rightarrow \quad \partial_t^\circ\,\vec\nu_i = -[\nabla_s\,\vec{\mathcal V}]^T\,\vec\nu_i \quad \text{on} \quad \Gamma_i(t)\,, \tag{3.16}$$

see [35, Lemma 9]. We also note that for $\vec\eta_i \in [H^1(\Gamma_i(t))]^d$ it holds that

$$\begin{aligned}\left[\frac{\delta}{\delta\Gamma}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\eta_i\right\rangle_{\Gamma_i(t)}\right](\vec\chi) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\eta_{i,\varepsilon}\right\rangle_{\Gamma_{i,\varepsilon}(t)}\mid_{\varepsilon=0} = \langle \nabla_s\,.\,\vec\eta_i, \nabla_s\,.\,\vec\chi\rangle_{\Gamma_i(t)} \\ &\quad + \sum_{l,\,m=1}^d \left[\langle (\vec\nu_i)_l\,(\vec\nu_i)_m\,\nabla_s\,(\vec\eta_i)_m, \nabla_s\,(\vec\chi)_l\rangle_{\Gamma_i(t)} - \langle (\nabla_s)_m\,(\vec\eta_i)_l, (\nabla_s)_l\,(\vec\chi)_m\rangle_{\Gamma_i(t)}\right] \\ &= \langle \nabla_s\,\vec\eta_i, \nabla_s\,\vec\chi\rangle_{\Gamma_i(t)} + \langle \nabla_s\,.\,\vec\eta_i, \nabla_s\,.\,\vec\chi\rangle_{\Gamma_i(t)} - \left\langle (\nabla_s\,\vec\eta_i)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\mathrm{id}})^T\right\rangle_{\Gamma_i(t)}\,,\end{aligned} \tag{3.17}$$

where $\partial_\varepsilon^0\,\vec\eta_i = \vec 0$, see Lemma 2 and the proof of Lemma 3 in [20]. Here

$$\underline{\underline{D}}(\vec\chi) := \nabla_s\,\vec\chi + (\nabla_s\,\vec\chi)^T\,,$$

and we note that our notation is such that $\nabla_s\,\vec\chi = (\nabla_\Gamma\,\vec\chi)^T$, with $\nabla_\Gamma\,\vec\chi = (\partial_{s_l}\,\chi_m)_{l,m=1}^d$ defined as in [20]. It follows from (3.17) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\eta\right\rangle_{\Gamma_i(t)} &= \left\langle \nabla_s\,\vec\eta, \nabla_s\,\vec{\mathcal V}\right\rangle_{\Gamma_i(t)} + \left\langle \nabla_s\,.\,\vec\eta, \nabla_s\,.\,\vec{\mathcal V}\right\rangle_{\Gamma_i(t)} \\ &\quad - \left\langle (\nabla_s\,\vec\eta)^T, \underline{\underline{D}}(\vec{\mathcal V})\,(\nabla_s\,\vec{\mathrm{id}})^T\right\rangle_{\Gamma_i(t)} \qquad \forall\ \vec\eta \in \{\vec\xi \in H^1(\Gamma_{i,T}) : \partial_t^\circ\,\vec\xi = \vec 0\}\,.\end{aligned} \tag{3.18}$$

Similarly to (3.13) it holds that

$$\left[\frac{\delta}{\delta\Gamma}\,\langle w, 1\rangle_{\gamma(t)}\right](\vec\chi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,\langle w_\varepsilon, 1\rangle_{\gamma_\varepsilon(t)}\mid_{\varepsilon=0} = \left\langle w\,\vec{\mathrm{id}}_s, \vec\chi_s\right\rangle_{\gamma(t)} \quad \forall\ w \in L^\infty(\gamma(t)),\ \vec\chi \in [H^1_\gamma(\Gamma(t))]^d\,, \tag{3.19}$$

where $\partial_\varepsilon^0\,w = 0$. Moreover, similarly to (3.17), we note that for $\vec\eta \in [H^1_\gamma(\Gamma(t))]^d$ it holds that

$$\left[\frac{\delta}{\delta\Gamma}\left\langle \vec{\mathrm{id}}_s, \vec\eta_s\right\rangle_{\gamma(t)}\right](\vec\chi) = \left\langle \underline{\underline{\mathcal P}}_\gamma\,\vec\eta_s, \vec\chi_s\right\rangle_{\gamma(t)}\,, \tag{3.20}$$

where $\partial_\varepsilon^0\,\vec\eta = \vec 0$, and where

$$\underline{\underline{\mathcal P}}_\gamma = \underline{\underline{\mathrm{Id}}} - \vec{\mathrm{id}}_s \otimes \vec{\mathrm{id}}_s \qquad \text{on} \ \ \gamma(t)\,. \tag{3.21}$$

Now combining (3.12a–d), on noting (3.13)–(3.21), yields that

$$f_\Gamma(\vec\chi) = \sum_{i=1}^{2}\Big[\langle\nabla_s\,\vec y_i, \nabla_s\,\vec\chi\rangle_{\Gamma_i(t)} + \langle\nabla_s\,.\,\vec y_i, \nabla_s\,.\,\vec\chi\rangle_{\Gamma_i(t)} - \Big\langle(\nabla_s\,\vec y_i)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T\Big\rangle_{\Gamma_i(t)}$$
$$-\tfrac12\Big\langle[\alpha_i\,|\vec\varkappa_i^\star - \overline\varkappa_i\,\vec\nu_i|^2 - 2\,(\vec\varkappa_i^\star\,.\,\underline{\underline{Q}}_{i,\theta}\,\vec y_i)]\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi\Big\rangle_{\Gamma_i(t)} + \alpha_i\,\overline\varkappa_i\,\Big\langle\vec\varkappa_i^\star, \partial_\varepsilon^0\,\vec\nu_i\Big\rangle_{\Gamma_i(t)}$$
$$+\Big\langle\partial_\varepsilon^0\,[\underline{\underline{Q}}_{i,\theta}\,\vec\varkappa_i^\star], \vec y_i\Big\rangle_{\Gamma_i(t)}\Big] - \varsigma\,\Big\langle\vec{\rm id}_s, \vec\chi_s\Big\rangle_{\gamma(t)}$$
$$+\Big\langle\vec\varkappa_\gamma^\star\,.\,\vec z - C_1\,(\vec{\rm m}_1 + \vec{\rm m}_2)\,.\,\vec\phi - \sum_{i=1}^{2}(\alpha_i^G\,\vec\varkappa_\gamma^\star + \vec y_i)\,.\,\vec{\rm m}_i, \vec{\rm id}_s\,.\,\vec\chi_s\Big\rangle_{\gamma(t)} + \Big\langle\underline{\underline{\mathcal{P}}}_\gamma\,\vec z_s, \vec\chi_s\Big\rangle_{\gamma(t)}$$
$$\forall\ \vec\chi\in[H^1_\gamma(\Gamma(t))]^d\,, \tag{3.22a}$$

$$\alpha_i\,(\vec\varkappa_i^\star - \overline\varkappa_i\,\vec\nu_i) - \underline{\underline{Q}}_{i,\theta}\,\vec y_i = \vec 0 \qquad \text{on}\ \ \Gamma_i(t)\,,\ \ i=1,2\,, \tag{3.22b}$$

$$\alpha_i^G\,\vec\varkappa_\gamma^\star + \vec y_i + C_1\,\vec\phi = \vec 0 \qquad \text{on}\ \ \gamma(t)\,,\ \ i=1,2\,, \tag{3.22c}$$

$$\sum_{i=1}^{2}\alpha_i^G\,\vec{\rm m}_i - \vec z = \vec 0 \qquad \text{on}\ \ \gamma(t)\,,\ \ i=1,2\,, \tag{3.22d}$$

with (3.7), (3.10) and (3.9). As $\partial_\varepsilon^0\,\vec\varkappa_i^\star = \vec 0$, we have that

$$\partial_\varepsilon^0\,[\underline{\underline{Q}}_{i,\theta}\,\vec\varkappa_i^\star] = (1-\theta)\,\Big[(\vec\varkappa_i^\star\,.\,\partial_\varepsilon^0\,\vec\nu_i)\,\vec\nu_i + (\vec\varkappa_i^\star\,.\,\vec\nu_i)\,\partial_\varepsilon^0\,\vec\nu_i\Big]\,. \tag{3.23}$$

We observe that (3.22b,c) imply that

$$\underline{\underline{Q}}_{i,\theta}\,\vec y_i = \alpha_i\,\vec\varkappa_i^\star - \alpha_i\,\overline\varkappa_i\,\vec\nu_i \quad \text{on}\ \ \Gamma_i(t) \qquad \text{and} \qquad \vec y_i + C_1\,\vec\phi = -\alpha_i^G\,\vec\varkappa_\gamma^\star \quad \text{on}\ \ \gamma(t)\,. \tag{3.24}$$

Let us now recover $\vec\varkappa_i^\star$ and $\vec\varkappa_\gamma^\star$ in terms of the geometry again. To this end, we first recall the identity

$$\int_{\Gamma_i(t)}\nabla_s\,g\,{\rm d}\mathcal{H}^{d-1} = -\int_{\Gamma_i(t)} g\,\varkappa_i\,\vec\nu_i\,{\rm d}\mathcal{H}^{d-1} + \int_{\gamma(t)} g\,\vec\mu_i\,{\rm d}\mathcal{H}^{d-2} \qquad \forall\ g\in H^1(\Gamma_i(t))\,, \tag{3.25}$$

see e.g. Theorem 2.10 in [21] and Proposition 4.5 in [36, p. 334]. It immediately follows from (3.7), (2.3) and (3.25) that $\vec{\rm m}_i = \vec\mu_i$ and $\underline{\underline{Q}}_{i,\theta}\,\vec\varkappa_i^\star = \vec\varkappa_i = \varkappa_i\,\vec\nu_i$, with the latter implying that

$$\vec\varkappa_i^\star\,.\,\vec\nu_i = \varkappa_i\,. \tag{3.26}$$

Hence we immediately get $\vec\varkappa_i^\star = \vec\varkappa_i$ for $\theta\in(0,1]$. For $\theta=0$, on the other hand, it follows from (3.24) and (3.26) that $\alpha_i\,\vec\varkappa_i^\star = [\vec y_i\,.\,\vec\nu_i + \alpha_i\,\overline\varkappa_i]\,\vec\nu_i$, and so $\vec\varkappa_i^\star = \varkappa_i\,\vec\nu_i = \vec\varkappa_i$. Moreover, combining (3.9) and (2.9) yields that $\vec\varkappa_\gamma^\star = \vec\varkappa_\gamma$. Overall, we obtain from (3.24) that

$$\underline{\underline{Q}}_{i,\theta}\,\vec y_i = \alpha_i\,(\varkappa_i - \overline\varkappa_i)\,\vec\nu_i \quad \text{on}\ \ \Gamma_i(t) \qquad \text{and} \qquad \vec y_i + C_1\,\vec\phi = -\alpha_i^G\,\vec\varkappa_\gamma \quad \text{on}\ \ \gamma(t)\,. \tag{3.27}$$

However, if $\theta\in(0,1]$, then the two conditions in (3.27) are incompatible in general if $\alpha_i^G\neq 0$, since the first condition in (3.27) yields that $\vec y_i = \alpha_i(\varkappa_i - \overline\varkappa_i)\,\vec\nu_i$. If $C_1=1$ then the two conditions are in general incompatible even if $\alpha_i^G = 0$. Hence for general boundaries $\gamma(t)$ and $\alpha_i^G\neq 0$ we need to take $\theta=0$, at least locally at the boundary. Therefore it may be desirable to consider a variable $\theta\in L^\infty(\Gamma(t))$. The calculation (3.22a–d) remains valid provided that $\partial_\varepsilon^0\,\theta = 0$. We will make this more rigorous on the discrete level, see (4.8) below.

Using (3.16), (3.23) and (3.22c,d) in (3.22a) yields the condensed version

$$f_\Gamma(\vec\chi) = \sum_{i=1}^{2}\Big[\langle\nabla_s\,\vec y_i, \nabla_s\,\vec\chi\rangle_{\Gamma_i(t)} + \langle\nabla_s\,.\,\vec y_i, \nabla_s\,.\,\vec\chi\rangle_{\Gamma_i(t)} - \Big\langle(\nabla_s\,\vec y_i)^T, \underline{\underline{D}}(\vec\chi)\,(\nabla_s\,\vec{\rm id})^T\Big\rangle_{\Gamma_i(t)}$$

$$-\tfrac{1}{2}\left\langle [\alpha_i\,|\vec\varkappa_i - \overline\varkappa_i\,\vec\nu_i|^2 - 2\,(\vec\varkappa_i\,.\,\underline{\underline{Q}}_{i,\theta}\,\vec y_i)]\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i(t)} - \alpha_i\,\overline\varkappa_i\left\langle \vec\varkappa_i, [\nabla_s\,\vec\chi]^T\,\vec\nu_i\right\rangle_{\Gamma_i(t)}$$
$$-(1-\theta)\left\langle\left[(\vec\varkappa_i\,.\,[\nabla_s\,\vec\chi]^T\,\vec\nu_i)\,\vec\nu_i + (\vec\varkappa_i\,.\,\vec\nu_i)\,[\nabla_s\,\vec\chi]^T\,\vec\nu_i\right], \vec y_i\right\rangle_{\Gamma_i(t)}\Bigg] - \varsigma\left\langle \vec{\rm id}_s, \vec\chi_s\right\rangle_{\gamma(t)}$$
$$+\sum_{i=1}^{2}\alpha_i^G\left[\left\langle \vec\varkappa_\gamma\,.\,\vec{\rm m}_i, \vec{\rm id}_s\,.\,\vec\chi_s\right\rangle_{\gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_\gamma\,(\vec{\rm m}_i)_s, \vec\chi_s\right\rangle_{\gamma(t)}\right] \qquad \forall\ \vec\chi\in[H^1_\gamma(\Gamma(t))]^d\,, \tag{3.28a}$$
$$\alpha_i\,(\vec\varkappa_i - \overline\varkappa_i\,\vec\nu_i) - \underline{\underline{Q}}_{i,\theta}\,\vec y_i = \vec 0 \qquad \text{on}\ \ \Gamma_i(t)\,,\ i=1,2\,, \tag{3.28b}$$
$$\alpha_i^G\,\vec\varkappa_\gamma + \vec y_i + C_1\,\vec\phi = \vec 0 \qquad \text{on}\ \ \gamma(t)\,,\ i=1,2\,, \tag{3.28c}$$
$$\left\langle \underline{\underline{Q}}_{i,\theta}\,\vec\varkappa_i, \vec\eta\right\rangle_{\Gamma_i(t)} + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta\right\rangle_{\Gamma_i(t)} = \langle \vec{\rm m}_i, \vec\eta\rangle_{\gamma(t)} \qquad \forall\ \vec\eta\in[H^1(\Gamma_i(t))]^d\,,\ i=1,2\,, \tag{3.28d}$$
$$C_1\,(\vec{\rm m}_1 + \vec{\rm m}_2) = \vec 0 \qquad \text{on}\quad \gamma(t)\,, \tag{3.28e}$$
$$\langle \vec\varkappa_\gamma, \vec\eta\rangle_{\gamma(t)} + \left\langle \vec{\rm id}_s, \vec\eta_s\right\rangle_{\gamma(t)} = 0 \quad \forall\ \vec\eta\in[H^1(\gamma(t))]^d\,. \tag{3.28f}$$

Remark 3.1. We recall from (3.27) and the discussion below that in general we require $\theta = 0$. If $C_1 = 0$ then it follows from (3.28c) that $\vec y_i = -\alpha_i^G\,\vec\varkappa_\gamma$ on $\gamma(t)$, for $i = 1, 2$. Combining this with (3.28b) for $\theta = 0$ then yields that (2.19a) holds.

On the other hand, in the case of a C^1–junction, when $C_1 = 1$, then (3.28e) implies that $\vec\mu_1 + \vec\mu_2 = \vec 0$ and hence that $\vec\nu_1 = \vec\nu_2 = \vec\nu$ on $\gamma(t)$, and so it follows from (3.28b,c) with $\theta = 0$ that

$$\alpha_i\,(\varkappa_i - \overline\varkappa_i) + \alpha_i^G\,\vec\varkappa_\gamma\,.\,\vec\nu + \vec\phi\,.\,\vec\nu = 0 \qquad \text{on}\ \ \gamma(t)\,,\ i=1,2\,,$$

which means that (2.20a) holds.

The weak formulation of a generalized L^2–gradient flow of $E((\Gamma_i(t))_{i=1}^2)$ can then be formulated as follows. Given $\Gamma_i(0)$, $i = 1, 2$, for all $t \in (0, T]$ find $\Gamma_i(t) = \vec x_i(\Upsilon_i, t)$, $i = 1, 2$, with $\vec{\mathcal{V}}(t) \in [H^1(\Gamma(t))]^d$, and $\vec\varkappa_i(t) \in [L^2(\Gamma_i(t))]^d$, $\vec y_i(t) \in [H^1(\Gamma_i(t))]^d$, $\vec{\rm m}_i(t) \in [H^1(\gamma(t))]^d$, $i = 1, 2$, as well as $\vec\varkappa_\gamma \in [L^2(\gamma(t))]^d$, $\vec z \in [L^2(\gamma(t))]^d$, $\vec\phi \in [L^2(\gamma(t))]^d$ such that

$$\left\langle \vec{\mathcal{V}}, \vec\chi\right\rangle_{\Gamma(t)} + \varrho\left\langle \vec{\mathcal{V}}, \vec\chi\right\rangle_{\gamma(t)} = f_\Gamma(\vec\chi) \quad \forall\ \vec\chi\in[H^1_\gamma(\Gamma(t))]^d \tag{3.29}$$

and (3.28a–f) hold. Here we note that $\varrho = 0$ recovers a weak formulation for the standard L^2–gradient flow. As stated in (1.2), we allow for $\varrho \geq 0$ in general, to allow for a damping of the movement of the contact line $\gamma(t)$. In numerical simulations such a damping often proves beneficial, as it suppresses possible oscillations at the contact line. On the other hand, such a dissipation mechanism at the boundary is probably also relevant in applications.

4. Semidiscrete finite element approximation

It is the aim of this section to introduce a semidiscrete continuous-in-time finite element approximation of the weak formulation (3.29), (3.28a–f) derived in the previous section. Our finite element discretization will be given by (4.27a–f) below, and the main result of this section is the stability proof in Theorem 4.1 below.

Similarly to [3], we introduce the following discrete spaces. Let $\Gamma^h(t) \subset \mathbb{R}^d$ be $(d-1)$-dimensional *polyhedral surfaces*, i.e. unions of non-degenerate $(d-1)$-simplices with no hanging vertices (see [17, p. 164] for $d = 3$), approximating the surfaces $\Gamma(t)$. In particular, let $\Gamma^h(t) = \bigcup_{j=1}^J \overline{\sigma_j^h(t)}$, where $\{\sigma_j^h(t)\}_{j=1}^J$ is a family mutually disjoint open $(d-1)$-simplices with vertices $\{\vec q_k^h(t)\}_{k=1}^K$. In analogy to the continuous setting, we write $\Gamma^h(t) = \Gamma_1^h(t) \cup \gamma^h(t) \cup \Gamma_2^h(t)$, where $\gamma^h(t) = \partial\Gamma_1^h(t) = \partial\Gamma_2^h(t)$. Here we let $\Gamma_i^h(t) = \bigcup_{j=1}^{J_i} \overline{\sigma_{i,j}^h(t)}$, with vertices $\{\vec{q}_{i,k}^h(t)\}_{k=1}^{K_i}$, $i = 1, 2$. We also assume that $\gamma^h(t)$ has the vertices $\{\vec{q}_{\gamma,k}^h(t)\}_{k=1}^{K_\gamma}$. Clearly, it holds that $J = J_1 + J_2$ and $K = K_1 + K_2 - K_\gamma$. Then let

$$\underline{V}^h(\Gamma_i^h(t)) = \{\vec{\chi} \in [C(\Gamma_i^h(t))]^d : \vec{\chi}\,|_{\sigma_{i,j}^h} \text{ is linear } \forall\ j = 1, \ldots, J_i\} = [W^h(\Gamma_i^h(t))]^d\,,\ i = 1, 2\,,$$

where $W^h(\Gamma_i^h(t))$ is the space of scalar continuous piecewise linear functions on $\Gamma_i^h(t)$, with $\{\chi_{i,k}^h(\cdot, t)\}_{k=1}^{K_i}$ denoting the standard basis of $W^h(\Gamma_i^h(t))$, i.e.

$$\chi_{i,k}^h(\vec{q}_{i,l}^h(t), t) = \delta_{kl} \qquad \forall\ k, l \in \{1, \ldots, K_i\}\,, t \in [0, T]\,. \tag{4.1}$$

In addition, let

$$\underline{V}^h(\Gamma^h(t)) = \{\vec{\chi} \in [C(\Gamma^h(t))]^d : \vec{\chi}\,|_{\Gamma_i^h(t)} \in \underline{V}^h(\Gamma_i^h(t)),\ i = 1, 2\} = [W^h(\Gamma^h(t))]^d\,.$$

We denote the basis functions of $W^h(\Gamma^h(t))$ by $\{\chi_k^h(\cdot, t)\}_{k=1}^{K}$. Moreover, let

$$\underline{V}^h(\gamma^h(t)) := \{\vec{\psi} \in [C(\gamma^h(t))]^d : \exists\ \vec{\chi} \in \underline{V}^h(\Gamma^h(t))\ \vec{\chi}\,|_{\gamma^h(t)} = \vec{\psi}\} =: [W^h(\gamma^h(t))]^d\,, \tag{4.2a}$$

$$\underline{V}_0^h(\Gamma^h(t)) := \{\vec{\chi} \in \underline{V}^h(\Gamma^h(t)) : \vec{\chi}\,|_{\gamma^h(t)} = \vec{0}\}\,, \tag{4.2b}$$

$$\underline{V}_0^h(\Gamma_i^h(t)) := \{\vec{\chi} \in \underline{V}^h(\Gamma_i^h(t)) : \vec{\chi}\,|_{\gamma^h(t)} = \vec{0}\}\,. \tag{4.2c}$$

We denote the basis functions of $W^h(\gamma^h(t))$ by $\{\phi_k^h(\cdot, t)\}_{k=1}^{K_\gamma}$. We require that $\Gamma_i^h(t) = \vec{X}^h(\Gamma_i^h(0), t)$ with $\vec{X}^h \in \underline{V}^h(\Gamma^h(0))$, and that $\vec{q}_k^h \in [H^1(0, T)]^d$, $k = 1, \ldots, K$.

We denote the L^2–inner products on $\Gamma^h(t)$, $\Gamma_i^h(t)$ and and $\gamma^h(t)$ by $\langle \cdot, \cdot \rangle_{\Gamma^h(t)}$, $\langle \cdot, \cdot \rangle_{\Gamma_i^h(t)}$ and $\langle \cdot, \cdot \rangle_{\gamma^h(t)}$, respectively. In addition, for piecewise continuous functions, with possible jumps across the edges of $\{\sigma_{i,j}^h\}_{j=1}^{J_i}$, we also introduce the mass lumped inner product

$$\langle \eta, \phi \rangle_{\Gamma_i^h(t)}^h = \sum_{j=1}^{J_i} \langle \eta, \phi \rangle_{\sigma_{i,j}^h(t)}^h := \sum_{j=1}^{J_i} \tfrac{1}{d}\, \mathcal{H}^{d-1}(\sigma_{i,j}^h(t)) \sum_{k=1}^{d} (\eta\, \phi)((\vec{q}_{i,j_k}^h(t))^-)\,,$$

where $\{\vec{q}_{i,j_k}^h(t)\}_{k=1}^d$ are the vertices of $\sigma_{i,j}^h(t)$, and where we define $\eta((\vec{q}_{i,j_k}^h(t))^-) := \lim_{\sigma_j^h(t) \ni \vec{p} \to \vec{q}_{i,j_k}^h(t)} \eta(\vec{p})$. We naturally extend this definition to vector and tensor functions. We also define the mass lumped inner products $\langle \cdot, \cdot \rangle_{\Gamma^h(t)}^h$ and $\langle \cdot, \cdot \rangle_{\gamma^h(t)}^h$ in the obvious way.

Let $\vec{\nu}_i^h$ denote the the outward unit normal to $\Gamma_i^h(t)$, $i = 1, 2$, and similarly let $\vec{\nu}^h$ denote the the outward unit normal to $\Gamma^h(t)$. Then we introduce the vertex normal functions $\vec{\omega}_i^h(\cdot, t) \in \underline{V}^h(\Gamma_i^h(t))$ with

$$\vec{\omega}_i^h(\vec{q}_{i,k}^h(t), t) := \frac{1}{\mathcal{H}^{d-1}(\Lambda_{i,k}^h(t))} \sum_{j \in \Theta_{i,k}^h} \mathcal{H}^{d-1}(\sigma_{i,j}^h(t))\, \vec{\nu}_i^h\,|_{\sigma_{i,j}^h(t)}\,, \tag{4.3}$$

where for $k = 1, \ldots, K_i$ we define $\Theta_{i,k}^h := \{j : \vec{q}_{i,k}^h(t) \in \overline{\sigma_{i,j}^h(t)}\}$ and set $\Lambda_{i,k}^h(t) := \cup_{j \in \Theta_{i,k}^h} \overline{\sigma_{i,j}^h(t)}$. Here we note that

$$\left\langle \vec{z}, w\, \vec{\nu}_i^h \right\rangle_{\Gamma_i^h(t)}^h = \left\langle \vec{z}, w\, \vec{\omega}_i^h \right\rangle_{\Gamma_i^h(t)}^h \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma_i^h(t))\,,\ w \in W^h(\Gamma_i^h(t))\,. \tag{4.4}$$

In the analogous fashion, we introduce the vertex normal function $\vec{\omega}^h(\cdot, t) \in \underline{V}^h(\Gamma^h(t))$, i.e. we set

$$\vec{\omega}^h(\vec{q}_k^h(t), t) := \frac{1}{\mathcal{H}^{d-1}(\Lambda_k^h(t))} \sum_{j \in \Theta_k^h} \mathcal{H}^{d-1}(\sigma_j^h(t))\, \vec{\nu}^h\,|_{\sigma_j^h(t)}\,, \tag{4.5}$$

where for $k = 1, \ldots, K$ we define $\Theta^h_k := \{j : \vec{q}^h_k(t) \in \overline{\sigma^h_j(t)}\}$ and set $\Lambda^h_k(t) := \cup_{j\in\Theta^h_k} \overline{\sigma^h_j(t)}$. Of course, it holds that

$$\left\langle \vec{z}, w\,\vec{\nu}^h \right\rangle^h_{\Gamma^h(t)} = \left\langle \vec{z}, w\,\vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^h(t))\,,\ w \in W^h(\Gamma^h(t))\,. \tag{4.6}$$

It clearly follows from (4.4) and (4.6) that

$$\left\langle \vec{z}, \vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} = \sum_{i=1}^2 \left\langle \vec{z}, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^h(t))\,. \tag{4.7}$$

In addition, for a given parameter $\theta \in [0,1]$ we introduce $\theta^h \in W^h(\Gamma^h(t))$ and $\theta^h_\star \in W^h(\Gamma^h(t))$ such that

$$\theta^h(\vec{q}^h_k(t), t) = \begin{cases} 0 & \vec{q}^h_k(t) \in \gamma^h(t)\,, \\ \theta & \vec{q}^h_k(t) \notin \gamma^h(t)\,, \end{cases} \quad \text{and} \quad \theta^h_\star(\vec{q}^h_k(t), t) = \begin{cases} 1 & \vec{q}^h_k(t) \in \gamma^h(t)\,, \\ \theta & \vec{q}^h_k(t) \notin \gamma^h(t)\,. \end{cases} \tag{4.8}$$

Then, similarly to (3.8), we introduce $\underline{\underline{Q}}^h_{i,\theta^h} \in [W^h(\Gamma^h_i(t))]^{d\times d}$ and $\underline{\underline{Q}}^h_{i,\theta^h_\star} \in [W^h(\Gamma^h_i(t))]^{d\times d}$ by setting, for $k \in \{1, \ldots, K_i\}$,

$$\underline{\underline{Q}}^h_{i,\theta^h}(\vec{q}^h_{i,k}(t), t) = \theta^h(\vec{q}^h_{i,k}(t), t)\,\underline{\underline{\rm Id}} + (1 - \theta^h(\vec{q}^h_{i,k}(t), t))\,\frac{\vec{\omega}^h_i(\vec{q}^h_{i,k}(t), t) \otimes \vec{\omega}^h_i(\vec{q}^h_{i,k}(t), t)}{|\vec{\omega}^h_i(\vec{q}^h_{i,k}(t), t)|^2}\,, \tag{4.9}$$

and similarly for $\underline{\underline{Q}}^h_{i,\theta^h_\star}$, where here and throughout we assume that $\vec{\omega}^h_i(\vec{q}^h_{i,k}(t), t) \neq \vec{0}$ for $k = 1, \ldots, K_i$ and $t \in [0,T]$. Only in pathological cases could this assumption be violated, and in practice this never occurred. We note that

$$\left\langle \underline{\underline{Q}}^h_{i,\theta^h}\,\vec{z}, \vec{v} \right\rangle^h_{\Gamma^h_i(t)} = \left\langle \vec{z}, \underline{\underline{Q}}^h_{i,\theta^h}\,\vec{v} \right\rangle^h_{\Gamma^h_i(t)} \quad \text{and} \quad \left\langle \underline{\underline{Q}}^h_{i,\theta^h}\vec{z}, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} = \left\langle \vec{z}, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} \tag{4.10}$$

for all $\vec{z}, \vec{v} \in \underline{V}^h(\Gamma^h_i(t))$, and analogously for θ^h in (4.10) replaced by $\theta^h_\star$. In addition, similarly to (4.7), it holds that

$$\sum_{i=1}^2 \left\langle \vec{z}, \underline{\underline{Q}}^h_{i,\theta^h_\star}\,\vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} = \sum_{i=1}^2 \left\langle \vec{z}, \underline{\underline{Q}}^h_{i,\theta^h_\star}\,\vec{\omega}^h \right\rangle^h_{\Gamma^h_i(t)} \qquad \forall\ \vec{z} \in \underline{V}^h(\Gamma^h(t))\,. \tag{4.11}$$

Following the approach in the continuous setting, recall (2.13), (3.7), (3.9), we consider the first variation of the discrete energy

$$\begin{aligned} E^h((\Gamma^h_i(t))_{i=1}^2) := \sum_{i=1}^2 \Big[\tfrac12\,\alpha_i \left\langle |\vec{\kappa}^h_i - \overline{\varkappa}_i\,\vec{\nu}^h_i|^2, 1 \right\rangle^h_{\Gamma^h_i(t)} + \alpha^G_i \Big[\left\langle \vec{\kappa}^h_\gamma, \vec{\rm m}^h_i \right\rangle^h_{\gamma^h(t)} + 2\,\pi\,m(\Gamma^h_i(t)) \Big] \Big] \\ + \varsigma\,\mathcal{H}^{d-2}(\gamma^h(t))\,, \end{aligned} \tag{4.12}$$

where $\vec{\kappa}^h_i \in \underline{V}^h(\Gamma^h_i(t))$, $\vec{\rm m}^h_i \in \underline{V}^h(\gamma^h(t))$, $i = 1,2$, and $\vec{\kappa}^h_\gamma \in \underline{V}^h(\gamma^h(t))$, subject to the side constraints

$$\left\langle \underline{\underline{Q}}^h_{i,\theta^h}\,\vec{\kappa}^h_i, \vec{\eta} \right\rangle^h_{\Gamma^h_i(t)} + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\eta} \right\rangle_{\Gamma^h_i(t)} = \left\langle \vec{\rm m}^h_i, \vec{\eta} \right\rangle^h_{\gamma^h(t)} \quad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma^h_i(t))\,,\ i = 1,2\,, \tag{4.13a}$$

$$\left\langle \vec{\kappa}^h_\gamma, \vec{\chi} \right\rangle^h_{\gamma^h(t)} + \left\langle \vec{\rm id}_s, \vec{\chi}_s \right\rangle_{\gamma^h(t)} = 0 \qquad \forall\ \vec{\chi} \in \underline{V}^h(\gamma^h(t))\,, \tag{4.13b}$$

$$C_1\,(\vec{\rm m}^h_1 + \vec{\rm m}^h_2) = \vec{0} \qquad \text{on} \quad \gamma^h(t)\,. \tag{4.13c}$$

In particular, we define the Lagrangian

$$\begin{aligned} L^h(t) = \sum_{i=1}^2 \Big[\tfrac12\,\alpha_i \left\langle |\vec{\kappa}^h_i - \overline{\varkappa}_i\,\vec{\nu}^h_i|^2, 1 \right\rangle^h_{\Gamma^h_i(t)} + \alpha^G_i \left\langle \vec{\kappa}^h_\gamma, \vec{\rm m}^h_i \right\rangle^h_{\gamma^h(t)} \Big] + \varsigma\,\mathcal{H}^{d-2}(\gamma^h(t)) - \left\langle \vec{\kappa}^h_\gamma, \vec{Z}^h \right\rangle^h_{\gamma^h(t)} \\ - \left\langle \vec{\rm id}_s, \vec{Z}^h_s \right\rangle_{\gamma^h(t)} - \sum_{i=1}^2 \Big[\left\langle (\underline{\underline{Q}}^h_{i,\theta^h}\,\vec{\kappa}^h_i, \vec{Y}^h_i \right\rangle^h_{\Gamma^h_i(t)} + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec{Y}^h_i \right\rangle_{\Gamma^h_i(t)} - \left\langle \vec{\rm m}^h_i, \vec{Y}^h_i \right\rangle^h_{\gamma^h(t)} \Big] \end{aligned}$$

$$+ C_1 \left\langle \vec{\rm m}_1^h + \vec{\rm m}_2^h, \vec\Phi^h \right\rangle_{\gamma^h(t)}^h , \tag{4.14}$$

where $\vec\kappa_i^h \in \underline{V}^h(\Gamma_i^h(t))$ and $\vec\kappa_\gamma^h \in \underline{V}^h(\gamma^h(t))$ satisfy (4.13a) and (4.13b), respectively, with $\vec Y_i^h \in \underline{V}^h(\Gamma_i^h(t))$ and $\vec Z^h \in \underline{V}^h(\gamma^h(t))$ being the corresponding Lagrange multipliers. Similarly, $\vec\Phi^h \in \underline{V}^h(\gamma^h(t))$ is a Lagrange multiplier for (4.13c). It turns out that when mimicking the continuous approach from Section 3 on the discrete level, we need several technical definitions to make the arguments rigorous. We present the majority of the necessary definitions and properties next, before proceeding with taking suitable variations of (4.14).

Following [21, (5.23)], we define the discrete material velocity for $\vec z \in \Gamma^h(t)$ by

$$\vec{\mathcal{V}}^h(\vec z, t) := \sum_{k=1}^K \left[\frac{\rm d}{{\rm d}t}\, \vec q_k^h(t)\right] \chi_k^h(\vec z, t)\,.$$

We also introduce the finite element spaces

$$W_T^h(\Gamma_{i,T}^h) := \{\phi \in C(\Gamma_{i,T}^h) : \phi(\cdot,t) \in W^h(\Gamma_i^h(t)) \quad \forall\, t \in [0,T]\,,\\ \phi(\vec q_{i,k}^h(\cdot),\cdot) \in H^1(0,T) \quad \forall\, k \in \{1,\ldots,K_i\}\}\,,$$

where $\Gamma_{i,T}^h := \bigcup_{t\in[0,T]} \Gamma_i^h(t) \times \{t\}$, as well as the vector valued analogue $\underline{V}_T^h(\Gamma_{i,T}^h)$. In a similar fashion, we introduce $W_T^h(\sigma_{j,T}^h)$ and $\underline{V}_T^h(\sigma_{j,T}^h)$ via e.g.

$$W_T^h(\sigma_{j,T}^h) := \{\phi \in C(\overline{\sigma_{j,T}^h}) : \phi(\cdot,t) \text{ is linear } \forall\, t \in [0,T]\,,\ \phi(\vec q_{j_k}^h(\cdot),\cdot) \in H^1(0,T) \quad k = 1,\ldots,d\}\,,$$

where $\{\vec q_{j_k}^h(t)\}_{k=1}^d$ are the vertices of $\sigma_j^h(t)$, and where $\sigma_{j,T}^h := \bigcup_{t\in[0,T]} \sigma_j^h(t) \times \{t\}$, for $j \in \{1,\ldots,J\}$. Moreover, we define the analogue variants $W_T^h(\Gamma_T^h)$ and $\underline{V}_T^h(\Gamma_T^h)$ on $\Gamma_T^h = \bigcup_{t\in[0,T]} \Gamma^h(t) \times \{t\}$, as well as $W_T^h(\gamma_T^h)$ and $\underline{V}_T^h(\gamma_T^h)$ on $\gamma_T^h := \bigcup_{t\in[0,T]} \gamma^h(t) \times \{t\}$, with the scalar space for the latter e.g. being given by

$$W_T^h(\gamma_T^h) := \{\psi \in C(\gamma_T^h) : \exists\, \chi \in W_T^h(\Gamma_T^h)\ \chi(\cdot,t)\,|_{\gamma^h(t)} = \psi(\cdot,t) \quad \forall\, t \in [0,T]\}\,.$$

Then, similarly to (3.1), we define the discrete material derivatives on $\Gamma^h(t)$ element-by-element via the equations

$$(\partial_t^{\circ,h}\, \phi)\,|_{\sigma_j^h(t)} = (\phi_t + \vec{\mathcal{V}}^h \,.\, \nabla\, \phi)\,|_{\sigma_j^h(t)} \qquad \forall\, \phi \in W_T^h(\sigma_{j,T}^h)\,,\ j \in \{1,\ldots,J\}\,.$$

On differentiating (4.1) with respect to t, it immediately follows that

$$\partial_t^{\circ,h}\, \chi_k^h = 0 \quad \forall\, k \in \{1,\ldots,K\}\,, \tag{4.15}$$

see also [21, Lem. 5.5]. It follows directly from (4.15) that

$$\partial_t^{\circ,h}\, \phi(\cdot,t) = \sum_{k=1}^K \chi_k^h(\cdot,t)\, \frac{\rm d}{{\rm d}t}\, \phi_k(t) \quad \text{on } \Gamma^h(t) \tag{4.16}$$

for $\phi(\cdot,t) = \sum_{k=1}^K \phi_k(t)\, \chi_k^h(\cdot,t) \in W^h(\Gamma^h(t))$.

We recall from [21, Lem. 5.6] that

$$\frac{\rm d}{{\rm d}t} \int_{\sigma_j^h(t)} \phi \;{\rm d}\mathcal{H}^{d-1} = \int_{\sigma_j^h(t)} \partial_t^{\circ,h}\, \phi + \phi\, \nabla_s \,.\, \vec{\mathcal{V}}^h \;{\rm d}\mathcal{H}^{d-1} \quad \forall\, \phi \in W_T^h(\sigma_{j,T}^h)\,, j \in \{1,\ldots,J\}\,. \tag{4.17}$$

Similarly, we recall from [7, Lem. 3.1] that

$$\frac{\rm d}{{\rm d}t} \langle \eta, \phi\rangle_{\sigma_j^h(t)}^h = \langle \partial_t^{\circ,h}\, \eta, \phi\rangle_{\sigma_j^h(t)}^h + \langle \eta, \partial_t^{\circ,h}\, \phi\rangle_{\sigma_j^h(t)}^h + \langle \eta\, \phi, \nabla_s \,.\, \vec{\mathcal{V}}^h\rangle_{\sigma_j^h(t)}^h \quad \forall\, \eta, \phi \in W_T^h(\sigma_{j,T}^h)\,, \tag{4.18}$$

for all $j \in \{1, \ldots, J\}$. Moreover, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \eta, \phi \rangle^h_{\gamma^h(t)} = \langle \partial_t^{\circ,h}\, \eta, \phi \rangle^h_{\gamma^h(t)} + \langle \eta, \partial_t^{\circ,h}\, \phi \rangle^h_{\gamma^h(t)} + \langle \eta\, \phi, \vec{\mathrm{id}}_s \,.\, \vec{\mathcal{V}}^h_s \rangle^h_{\gamma^h(t)} \;\forall\, \eta, \phi \in W^h_T(\gamma^h_T)\,. \tag{4.19}$$

We also note the discrete version of the time derivative variant of (3.17),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s\, \vec\eta \right\rangle_{\Gamma^h_i(t)} &= \left\langle \nabla_s\, \vec\eta, \nabla_s\, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h_i(t)} + \left\langle \nabla_s \,.\, \vec\eta, \nabla_s \,.\, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h_i(t)} \\ &\quad - \left\langle (\nabla_s\, \vec\eta)^T, \underline{\underline{D}}(\vec{\mathcal{V}}^h)\, (\nabla_s \,\vec{\mathrm{id}})^T \right\rangle_{\Gamma^h_i(t)} \quad \forall\, \vec\eta \in \{\vec\xi \in \underline{V}^h_T(\Gamma^h_{i,T}) : \partial_t^{\circ,h}\, \vec\xi = \vec 0\}\,, \end{aligned} \tag{4.20}$$

as well as the corresponding version for $\gamma^h(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \vec{\mathrm{id}}_s, \vec\eta_s \right\rangle_{\gamma^h(t)} = \left\langle \underline{\underline{\mathcal{P}}}^h_\gamma\, \vec\eta_s, \vec{\mathcal{V}}^h_s \right\rangle_{\gamma^h(t)} \quad \forall\, \vec\eta \in \{\vec\xi \in \underline{V}^h_T(\gamma^h_T) : \partial_t^{\circ,h}\, \vec\xi = \vec 0\}\,, \tag{4.21}$$

which follows similarly to (3.20). Here, similarly to (3.21), we have defined

$$\underline{\underline{\mathcal{P}}}^h_\gamma = \underline{\underline{\mathrm{Id}}} - \vec{\mathrm{id}}_s \otimes \vec{\mathrm{id}}_s \quad \text{on} \quad \gamma^h(t)\,. \tag{4.22}$$

Finally, when taking variations of (4.14), we need to compute variations of the discrete vertex normals $\vec\omega^h_i$. To this end, for any given $\vec\chi \in \underline{V}^h(\Gamma^h(t))$ we introduce $\Gamma^h_\varepsilon(t)$ as in (3.5) and $\partial^{0,h}_\varepsilon$ defined by (3.14), both with $\Gamma(t)$ replaced by $\Gamma^h(t)$. We then observe that it follows from (4.4) with $w = 1$ and the discrete analogue of (3.15) that

$$\begin{aligned}\left\langle \vec z, \partial^{0,h}_\varepsilon\, \vec\omega^h_i \right\rangle^h_{\Gamma^h_i(t)} &= \left\langle \vec z, \partial^{0,h}_\varepsilon\, \vec\nu^h_i \right\rangle^h_{\Gamma^h_i(t)} + \left\langle (\vec z \,.\, (\vec\nu^h_i - \vec\omega^h_i))\, \nabla_s \,\vec{\mathrm{id}}, \nabla_s\, \vec\chi \right\rangle^h_{\Gamma^h_i(t)} \\ &\qquad \forall\, \vec z \in \underline{V}^h(\Gamma^h_i(t)),\, \vec\chi \in \underline{V}^h(\Gamma^h(t))\,, \end{aligned} \tag{4.23}$$

where $\partial^{0,h}_\varepsilon\, \vec z = \vec 0$. In addition, we note that for all $\vec\xi$, $\vec\eta \in \underline{V}^h(\Gamma^h_i(t))$ with $\partial^{0,h}_\varepsilon\, \vec\xi = \partial^{0,h}_\varepsilon\, \vec\eta = \vec 0$ it holds that

$$\partial^{0,h}_\varepsilon\, \pi^h_i \left[\left(\vec\xi \,.\, \frac{\vec\omega^h_i}{|\vec\omega^h_i|} \right) \left(\vec\eta \,.\, \frac{\vec\omega^h_i}{|\vec\omega^h_i|} \right) \right] = \pi^h_i \left[\vec G^h_i(\vec\xi, \vec\eta) \,.\, \partial^{0,h}_\varepsilon\, \vec\omega^h_i \right] \quad \text{on } \Gamma^h_i(t)\,, \tag{4.24}$$

where

$$\vec G^h_i(\vec\xi, \vec\eta) = \frac{1}{|\vec\omega^h_i|^2} \left((\vec\xi \,.\, \vec\omega^h_i)\, \vec\eta + (\vec\eta \,.\, \vec\omega^h_i)\, \vec\xi - 2\, \frac{(\vec\eta \,.\, \vec\omega^h_i)\, (\vec\xi \,.\, \vec\omega^h_i)}{|\vec\omega^h_i|^2}\, \vec\omega^h_i \right), \tag{4.25}$$

and where $\pi^h_i(t) : C(\Gamma^h_i(t)) \to W^h(\Gamma^h_i(t))$ is the standard interpolation operator at the nodes $\{\vec q^h_{i,k}(t)\}_{k=1}^{K_i}$. It follows that

$$\vec G^h_i(\vec\xi, \vec\eta) \,.\, \vec\omega^h_i = 0 \quad \forall\, \vec\xi,\; \vec\eta \in \underline{V}^h(\Gamma^h_i(t))\,. \tag{4.26}$$

We are now in a position to formally derive the L^2–gradient flow of $E^h(t)$ subject to the side constraints (4.13a–c). In particular, on recalling the formal calculus of PDE constrained optimization, we set $[\frac{\delta}{\delta \Gamma^h}\, L^h](\vec\chi) = -\sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\, \vec{\mathcal{V}}^h, \vec\chi \right\rangle^h_{\Gamma^h_i(t)}$ for $\vec\chi \in \underline{V}^h(\Gamma^h(t))$, $[\frac{\delta}{\delta \vec\kappa^h_i}\, L^h](\vec\xi) = 0$ for $\vec\xi \in \underline{V}^h(\Gamma^h_i(t))$, $[\frac{\delta}{\delta \vec Y^h_i}\, L^h](\vec\eta) = 0$ for $\vec\eta \in \underline{V}^h(\Gamma^h_i(t))$, $[\frac{\delta}{\delta \vec{\mathrm{m}}^h_i}\, L^h](\vec\varphi) = 0$ for $\vec\varphi \in \underline{V}^h(\gamma^h(t))$, $[\frac{\delta}{\delta \vec\kappa^h_\gamma}\, L^h](\vec\phi) = 0$ for $\vec\phi \in \underline{V}^h(\gamma^h(t))$, leading to $\vec Z^h = \sum_{i=1}^2 \alpha^G_i\, \vec{\mathrm{m}}^h_i$, $[\frac{\delta}{\delta \vec Z^h}\, L^h](\vec\phi) = 0$ for $\vec\phi \in \underline{V}^h(\gamma^h(t))$ and $[\frac{\delta}{\delta \vec\Phi^h}\, L^h](\vec\eta) = 0$ for $\vec\eta \in \underline{V}^h(\gamma^h(t))$. Here we recall the definition of $\theta^h_\star$ in (4.8). We employ this doctored version of θ^h in order to obtain existence and uniqueness for the fully discrete approximation introduced in the next section. See also Remark 3.1 in [6], where the analogue to our situation here corresponds to two curves meeting in the plane, i.e. $N = d = 2$ in their notation.

Overall this gives rise to the following semidiscrete finite element approximation of the gradient flow (3.29), where we have noted the discrete version of (3.16), (4.23), (4.24), (4.26), variational versions of (4.18)–(4.21) and that $\partial_\varepsilon^{0,h}\,\theta^h = 0$. Given $\Gamma^h(0)$, find $(\Gamma^h(t))_{t\in(0,T]}$ such that $\vec{\rm id}\,|_{\Gamma^h(\cdot)}\in \underline{V}^h_T(\Gamma^h_T)$. In addition, for all $t\in(0,T]$ find $(\vec\kappa^h_i,\vec Y^h_i)\in[\underline{V}^h(\Gamma^h_i(t))]^2$, $i=1,2$, $\vec\kappa^h_\gamma\in\underline{V}^h(\gamma^h(t))$, $\vec{\rm m}^h_i\in\underline{V}^h(\gamma^h(t))$, $i=1,2$, and $C_1\,\Phi^h\in\underline{V}^h(\gamma^h(t))$ such that

$$
\begin{aligned}
&\sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\,\vec{\mathcal V}^h,\vec\chi\right\rangle^h_{\Gamma^h_i(t)} + \varrho\left\langle\vec{\mathcal V}^h,\vec\chi\right\rangle^h_{\gamma^h(t)}\\
&= \sum_{i=1}^2\Big[\left\langle\nabla_s\vec Y^h_i,\nabla_s\vec\chi\right\rangle_{\Gamma^h_i(t)} + \left\langle\nabla_s\,.\,\vec Y^h_i,\nabla_s\,.\,\vec\chi\right\rangle_{\Gamma^h_i(t)} - \left\langle(\nabla_s\vec Y^h_i)^T,\underline{\underline{D}}(\vec\chi)\,(\nabla_s\vec{\rm id})^T\right\rangle_{\Gamma^h_i(t)}\\
&\quad -\tfrac12\left\langle[\alpha_i\,|\vec\kappa^h_i-\overline{\varkappa}_i\,\vec\nu^h_i|^2 - 2\,(\vec Y^h_i\,.\,\underline{\underline{Q}}^h_{i,\theta^h}\,\vec\kappa^h_i)]\,\nabla_s\vec{\rm id},\nabla_s\vec\chi\right\rangle^h_{\Gamma^h_i(t)}\\
&\quad -\alpha_i\,\overline{\varkappa}_i\left\langle\vec\kappa^h_i,[\nabla_s\vec\chi]^T\,\vec\nu^h_i\right\rangle^h_{\Gamma^h_i(t)} + \left\langle(1-\theta^h)\,(\vec G^h_i(\vec Y^h_i,\vec\kappa^h_i)\,.\,\vec\nu^h_i)\,\nabla_s\vec{\rm id},\nabla_s\vec\chi\right\rangle^h_{\Gamma^h_i(t)}\\
&\quad -\left\langle(1-\theta^h)\,\vec G^h_i(\vec Y^h_i,\vec\kappa^h_i),[\nabla_s\vec\chi]^T\,\vec\nu^h_i\right\rangle^h_{\Gamma^h_i(t)}\Big]\\
&\quad +\sum_{i=1}^2\alpha^G_i\left[\left\langle\vec\kappa^h_\gamma\,.\,\vec{\rm m}^h_i,\vec{\rm id}_s\,.\,\vec\chi_s\right\rangle^h_{\gamma^h(t)} + \left\langle\underline{\underline{\mathcal P}}^h_\gamma\,(\vec{\rm m}^h_i)_s,\vec\chi_s\right\rangle_{\gamma^h(t)}\right] - \varsigma\left\langle\vec{\rm id}_s,\vec\chi_s\right\rangle_{\gamma^h(t)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall\ \vec\chi\in\underline{V}^h(\Gamma^h(t))\,,
\end{aligned}
\tag{4.27a}
$$

$$
\left\langle\underline{\underline{Q}}^h_{i,\theta^h}\,\vec\kappa^h_i,\vec\eta\right\rangle^h_{\Gamma^h_i(t)} + \left\langle\nabla_s\vec{\rm id},\nabla_s\vec\eta\right\rangle_{\Gamma^h_i(t)} = \left\langle\vec{\rm m}^h_i,\vec\eta\right\rangle^h_{\gamma^h(t)}\quad\forall\ \vec\eta\in\underline{V}^h(\Gamma^h_i(t))\,,\ i=1,2\,,\tag{4.27b}
$$

$$
\left\langle\vec\kappa^h_\gamma,\vec\chi\right\rangle^h_{\gamma^h(t)} + \left\langle\vec{\rm id}_s,\vec\chi_s\right\rangle_{\gamma^h(t)} = 0\qquad\forall\ \vec\chi\in\underline{V}^h(\gamma^h(t))\,,\tag{4.27c}
$$

$$
C_1\,(\vec{\rm m}^h_1+\vec{\rm m}^h_2) = \vec 0\qquad\text{on}\quad\gamma^h(t)\,,\tag{4.27d}
$$

$$
\alpha^G_i\,\vec\kappa^h_\gamma + \vec Y^h_i + C_1\,\vec\Phi^h = \vec 0\qquad\text{on}\quad\gamma^h(t)\,,\ i=1,2\,,\tag{4.27e}
$$

$$
\left\langle\alpha_i\,(\vec\kappa^h_i-\overline{\varkappa}_i\,\vec\nu^h_i) - \underline{\underline{Q}}^h_{i,\theta^h}\,\vec Y^h_i,\vec\xi\right\rangle^h_{\Gamma^h_i(t)} = 0\qquad\forall\ \vec\xi\in\underline{V}^h(\Gamma^h_i(t))\,,\ i=1,2\,.\tag{4.27f}
$$

We observe that choosing $\vec\xi = \alpha_i^{-1}\,\vec\pi^h_i[\underline{\underline{Q}}^h_{i,\theta^h}\,\vec\eta]$ in (4.27f) and combining with (4.27b), on recalling (4.10) and (4.4), yields that

$$
\alpha_i^{-1}\left\langle\underline{\underline{Q}}^h_{i,\theta^h}\,\vec Y^h_i,\underline{\underline{Q}}^h_{i,\theta^h}\,\vec\eta\right\rangle^h_{\Gamma^h_i(t)} + \left\langle\nabla_s\vec{\rm id},\nabla_s\vec\eta\right\rangle_{\Gamma^h_i(t)} = \left\langle\vec{\rm m}^h_i,\vec\eta\right\rangle^h_{\gamma^h(t)} - \overline{\varkappa}_i\left\langle\vec\omega^h_i,\vec\eta\right\rangle^h_{\Gamma^h_i(t)}\quad\forall\ \vec\eta\in\underline{V}^h(\Gamma^h_i(t))\,.\tag{4.28}
$$

Here $\vec\pi^h_i(t):[C(\Gamma^h_i(t))]^d\to\underline{V}^h(\Gamma^h_i(t))$ is the standard interpolation operator at the nodes $\{\vec q^h_{i,k}(t)\}_{k=1}^{K_i}$.

In order to be able to consider area and volume preserving variants of (4.27a–f), we introduce the Lagrange multipliers $\lambda^{A,h}_i(t)\in\mathbb R$, $i=1,2$, and $\lambda^{V,h}(t)\in\mathbb R$ for the constraints

$$
\frac{\rm d}{{\rm d}t}\,\mathcal H^{d-1}(\Gamma^h_i(t)) = \left\langle\nabla_s\,.\,\vec{\mathcal V}^h,1\right\rangle_{\Gamma^h_i(t)} = \left\langle\nabla_s\vec{\rm id},\nabla_s\vec{\mathcal V}^h\right\rangle_{\Gamma^h_i(t)} = 0\,,\tag{4.29}
$$

where we recall (4.17), and

$$
\frac{\rm d}{{\rm d}t}\,\mathcal L^d(\Omega^h(t)) = \left\langle\vec{\mathcal V}^h,\vec\nu^h\right\rangle_{\Gamma^h(t)} = \left\langle\vec{\mathcal V}^h,\vec\omega^h\right\rangle^h_{\Gamma^h(t)} = 0\,,\tag{4.30}
$$

where we note a discrete variant of (3.4) and (4.6). Here $\Omega^h(t)$ denotes the interior of $\Gamma^h(t)$. On recalling (4.7), (4.10) and (4.11), we can rewrite the constraint (4.30) as

$$0 = \left\langle \vec{\mathcal{V}}^h, \vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} = \sum_{i=1}^2 \left\langle \vec{\mathcal{V}}^h, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} = \sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\, \vec{\mathcal{V}}^h, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} = \sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\, \vec{\mathcal{V}}^h, \vec{\omega}^h \right\rangle^h_{\Gamma^h_i(t)} . \qquad (4.31)$$

Hence, on writing (4.27a) as

$$\sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\, \vec{\mathcal{V}}^h, \vec{\chi} \right\rangle^h_{\Gamma^h_i(t)} + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\chi} \right\rangle^h_{\gamma^h(t)} = \left\langle \vec{r}^h, \vec{\chi} \right\rangle^h_{\Gamma^h(t)} ,$$

we consider

$$\sum_{i=1}^2 \left\langle \underline{\underline{Q}}^h_{i,\theta^h_\star}\, \vec{\mathcal{V}}^h, \vec{\chi} \right\rangle^h_{\Gamma^h_i(t)} + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\chi} \right\rangle^h_{\gamma^h(t)} = \left\langle \vec{r}^h, \vec{\chi} \right\rangle^h_{\Gamma^h(t)} - \lambda^{V,h} \left\langle \vec{\omega}^h, \vec{\chi} \right\rangle^h_{\Gamma^h(t)} - \sum_{i=1}^2 \lambda^{A,h}_i \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\chi} \right\rangle_{\Gamma^h_i(t)} \qquad (4.32)$$

for all $\vec{\chi} \in \underline{V}^h(\Gamma^h(t))$, where $\lambda^{V,h}(t) \in \mathbb{R}$ and $\lambda^{A,h}_i(t) \in \mathbb{R}$, $i = 1, 2$, need to be determined. Of course, if we consider a volume preserving variant only, then we let $\lambda^{A,h}_1(t) = \lambda^{A,h}_2(t) = 0$ and

$$\lambda^{V,h}(t) = \left[\left\langle \vec{r}^h, \vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} - \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\omega}^h \right\rangle^h_{\gamma^h(t)} \right] / \left\langle \vec{\omega}^h, \vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} , \qquad (4.33)$$

which we derived on choosing $\vec{\chi} = \vec{\omega}^h$ in (4.32), and noting (4.31).

For the general volume and area preserving flow, we introduce the projection $\vec{\Pi}^h_0 : \underline{V}^h(\Gamma^h(t)) \to \underline{V}^h_0(\Gamma^h(t))$ onto $\underline{V}^h_0(\Gamma^h(t))$, recall (4.2b), and similarly $\vec{\Pi}^h_{i,0} : \underline{V}^h(\Gamma^h_i(t)) \to \underline{V}^h_0(\Gamma^h_i(t))$. We introduce the symmetric bilinear forms $a^h_{i,\theta} : \underline{V}^h(\Gamma^h_i(t)) \times \underline{V}^h(\Gamma^h_i(t)) \to \mathbb{R}$ by setting

$$a^h_{i,\theta}(\vec{\zeta}, \vec{\eta}) = \left\langle \underline{\underline{Q}}^h_{i,\theta^h}\, \vec{\zeta}, \vec{\Pi}^h_{i,0}\, \vec{\eta} \right\rangle^h_{\Gamma^h_i(t)} \qquad \forall\ \vec{\zeta}, \vec{\eta} \in \underline{V}^h(\Gamma^h_i(t)) ,\ i = 1, 2 , \qquad (4.34)$$

where we have noted (4.10). It holds that $a^h_{i,\theta}(\vec{\zeta}, \vec{\zeta}) \geq 0$ for all $\vec{\zeta} \in \underline{V}^h(\Gamma^h_i(t))$, with the inequality being strict if $\vec{\Pi}^h_{i,0}\,[\vec{Q}^h_{i,\theta^h}\, \vec{\zeta}] \neq \vec{0}$. Hence the Cauchy–Schwarz inequality holds, i.e.

$$|a^h_{i,\theta}(\vec{\zeta}, \vec{\eta})| \leq [a^h_{i,\theta}(\vec{\zeta}, \vec{\zeta})]^{\frac{1}{2}}\, [a^h_{i,\theta}(\vec{\eta}, \vec{\eta})]^{\frac{1}{2}} \qquad \forall\ \vec{\zeta}, \vec{\eta} \in \underline{V}^h(\Gamma^h_i(t)) ,\ i = 1, 2 , \qquad (4.35)$$

with strict inequality if $\vec{\Pi}^h_{i,0}\,[\vec{Q}^h_{i,\theta^h}\, \vec{\zeta}]$ and $\vec{\Pi}^h_{i,0}\,[\vec{Q}^h_{i,\theta^h}\, \vec{\eta}]$ are linearly independent. Then we note, on recalling (4.7), (4.27b) and (4.10), that

$$- \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\Pi}^h_0\, \vec{\omega}^h \right\rangle_{\Gamma^h_i(t)} = - \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\Pi}^h_{i,0}\, \vec{\omega}^h_i \right\rangle_{\Gamma^h_i(t)} = a^h_{i,\theta}(\vec{\kappa}^h_i, \vec{\omega}^h_i) = \left\langle \vec{\omega}^h_i, \vec{\Pi}^h_{i,0}\, \vec{\kappa}^h_i \right\rangle^h_{\Gamma^h_i(t)} \qquad (4.36a)$$

and

$$- \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\Pi}^h_{i,0}\, \vec{\kappa}^h_i \right\rangle_{\Gamma^h_i(t)} = a^h_{i,\theta}(\vec{\kappa}^h_i, \vec{\kappa}^h_i) . \qquad (4.36b)$$

In addition, it follows from (4.7), (4.10) and (4.34) that

$$\left\langle \vec{\omega}^h, \vec{\Pi}^h_0\, \vec{\omega}^h \right\rangle^h_{\Gamma^h(t)} = \sum_{i=1}^2 \left\langle \vec{\omega}^h_i, \vec{\Pi}^h_0\, \vec{\omega}^h \right\rangle^h_{\Gamma^h_i(t)} = \sum_{i=1}^2 \left\langle \vec{\omega}^h_i, \vec{\Pi}^h_{i,0}\, \vec{\omega}^h_i \right\rangle^h_{\Gamma^h_i(t)} = \sum_{i=1}^2 a^h_{i,\theta}(\vec{\omega}^h_i, \vec{\omega}^h_i) . \qquad (4.37)$$

Then (4.32), (4.37) and (4.36a,b) yield that $(\lambda^{V,h}, \lambda^{A,h}_1, \lambda^{A,h}_2)(t)$ are such that

$$\begin{pmatrix} \sum_{i=1}^2 a^h_{i,\theta}(\vec{\omega}^h_i, \vec{\omega}^h_i) & a^h_{1,\theta}(\vec{\kappa}^h_1, \vec{\omega}^h_1) & a^h_{2,\theta}(\vec{\kappa}^h_2, \vec{\omega}^h_2) \\ a^h_{1,\theta}(\vec{\kappa}^h_1, \vec{\omega}^h_1) & a^h_{1,\theta}(\vec{\kappa}^h_1, \vec{\kappa}^h_1) & 0 \\ a^h_{2,\theta}(\vec{\kappa}^h_2, \vec{\omega}^h_2) & 0 & a^h_{2,\theta}(\vec{\kappa}^h_2, \vec{\kappa}^h_2) \end{pmatrix} \begin{pmatrix} -\lambda^{V,h}(t) \\ \lambda^{A,h}_1(t) \\ \lambda^{A,h}_2(t) \end{pmatrix} = \begin{pmatrix} b_0(t) \\ b_1(t) \\ b_2(t) \end{pmatrix} , \qquad (4.38a)$$

where

$$b_0(t) = \sum_{i=1}^{2} \left\langle \vec\Pi_0^h \,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h - \left\langle \vec{r}^h, \vec\Pi_0^h \,\vec\omega^h \right\rangle_{\Gamma^h(t)}^h, \tag{4.38b}$$

$$b_i(t) = \left\langle \vec\Pi_{i,0}^h \,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \vec{\mathrm{m}}_i^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h - \left\langle \vec{r}^h, \vec\Pi_{i,0}^h \,\vec\kappa_i^h \right\rangle_{\Gamma^h(t)}^h, \ i = 1,2\,. \tag{4.38c}$$

On recalling (4.35), we observe that the matrix in (4.38a) is symmetric and positive definite as long as $\vec\Pi_{i,0}^h \,\vec\omega_i^h$ and $\vec\Pi_{i,0}^h\,[\vec{Q}_{i,\theta^h}^h \,\vec\kappa_i^h]$ are linearly independent, for $i = 1, 2$. The right hand sides (4.38b,c) are obtained by recalling (4.32), and on noting that (4.10) and (4.31) imply that

$$\begin{aligned}\sum_{i=1}^{2} \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h \,\vec{\mathcal{V}}^h, \vec\Pi_0^h \,\vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h &= \sum_{i=1}^{2} \left[\left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h \,\vec{\mathcal{V}}^h, \vec\Pi_0^h \,\vec\omega^h - \vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h \,\vec{\mathcal{V}}^h, \vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h \right] \\ &= \sum_{i=1}^{2} \left\langle \vec{\mathcal{V}}^h, \vec\Pi_0^h \,\vec\omega^h - \vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h = \sum_{i=1}^{2} \left\langle \vec\Pi_0^h \,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \vec\omega^h \right\rangle_{\Gamma_i^h(t)}^h,\end{aligned} \tag{4.39}$$

while (4.8), (4.10), (4.27b) and (4.29) yield that

$$\begin{aligned}\left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h \,\vec{\mathcal{V}}^h, \vec\Pi_{i,0}^h \,\vec\kappa_i^h \right\rangle_{\Gamma_i^h(t)}^h &= \left\langle \vec\Pi_{i,0}^h \,\vec{\mathcal{V}}^h, \underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h \right\rangle_{\Gamma_i^h(t)}^h \\ &= \left\langle \vec\Pi_{i,0}^h \,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \vec{\mathrm{m}}_i^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h - \left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)} \\ &= \left\langle \vec\Pi_{i,0}^h \,\vec{\mathcal{V}}^h - \vec{\mathcal{V}}^h, \underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \vec{\mathrm{m}}_i^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h.\end{aligned} \tag{4.40}$$

We see that on removing the last two rows and columns in (4.38a), we obtain an expression similar to (4.33) for $\lambda^{V,h}(t)$, but here we test with $\vec\Pi_0^h \,\vec\omega^h$ as opposed to $\vec\omega^h$. Analogously, if we want to consider phase area preservations only, then removing the first row and column in (4.38a) yields a reduced system for the two Lagrange multipliers $\lambda_i^{A,h}(t)$, $i = 1, 2$.

The following theorem establishes that (4.27a–f) is indeed a weak formulation for a generalized L^2–gradient flow of $E^h(t)$ subject to the side constraints (4.13a–c). We will also show that for $\theta = 0$ the scheme produces *conformal polyhedral surfaces* $\Gamma_1(t)$ and $\Gamma_2(t)$. Here we recall from [10], see also [3, §4.1], that the open surfaces $\Gamma_i^h(t)$, $i = 1, 2$, are conformal polyhedral surfaces if

$$\left\langle \nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec\eta \right\rangle_{\Gamma_i^h(t)} = 0 \quad \forall\ \vec\eta \in \left\{ \vec\xi \in \underline{V}_0^h(\Gamma_i^h(t)) : \vec\xi(\vec{q}_{i,k}^h(t))\,.\,\vec\omega_i^h(\vec{q}_{i,k}^h(t),t) = 0,\ k = 1,\ldots,K_i \right\},\ i = 1,2\,. \tag{4.41}$$

We recall from [3, 10] that conformal polyhedral surfaces exhibit good meshes. Moreover, we recall that in the case $d = 2$, conformal polyhedral surfaces are equidistributed polygonal curves, see [2, 5].

Theorem 4.1. *Let* $\theta \in [0, 1]$, $\varrho \geq 0$ *and let* $\{(\Gamma^h, \vec\kappa_1^h, \vec\kappa_2^h, \vec{Y}_1^h, \vec{Y}_2^h, \vec\kappa_\gamma^h, \vec{\mathrm{m}}_1^h, \vec{\mathrm{m}}_2^h, \vec\Phi^h)(t)\}_{t\in[0,T]}$ *be a solution to* (4.27a–f). *In addition, we assume that* $\vec\kappa_\gamma^h \in \underline{V}_T^h(\gamma_T^h)$, $\vec\kappa_i^h, \vec\pi_i^h[\underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h] \in \underline{V}_T^h(\Gamma_{i,T}^h)$, $\vec{\mathrm{m}}_i^h \in \underline{V}_T^h(\gamma_T^h)$, $i = 1, 2$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h((\Gamma_i^h(t))_{i=1}^2) = -\sum_{i=1}^{2} \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h \,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h - \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h. \tag{4.42}$$

Moreover, if $\theta = 0$ *then* $\Gamma_1^h(t)$ *and* $\Gamma_2^h(t)$ *are open conformal polyhedral surfaces for all* $t \in (0, T]$.

Proof. Taking the time derivative of (4.13a), where we choose discrete test functions $\vec\eta$ such that $\partial_t^{\circ,h}\,\vec\eta = \vec{0}$, yields for $i = 1, 2$ that

$$\left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h), \vec\eta \right\rangle_{\Gamma_i^h(t)}^h + \left\langle [(\underline{\underline{Q}}_{i,\theta^h}^h \,\vec\kappa_i^h)\,.\,\vec\eta]\,\nabla_s \,\vec{\mathrm{id}}, \nabla_s \,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \nabla_s \,\vec{\mathcal{V}}^h, \nabla_s \,\vec\eta \right\rangle_{\Gamma_i^h(t)}$$

$$+\left\langle \nabla_s \,.\, \vec{\mathcal{V}}^h, \nabla_s \,.\, \vec\eta \right\rangle_{\Gamma_i^h(t)} - \left\langle (\nabla_s\,\vec\eta)^T, \underline{\underline{D}}(\vec{\mathcal{V}}^h)\,(\nabla_s\,\vec{\rm id})^T \right\rangle_{\Gamma_i^h(t)} = \left\langle \partial_t^{\circ,h}\,\vec{\rm m}_i^h, \vec\eta \right\rangle_{\gamma^h(t)}^h + \left\langle \vec{\rm m}_i^h \,.\, \vec\eta, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)}^h, \tag{4.43}$$

where we have noted (4.18), (4.19), (4.20) and that $\vec\pi_i^h[\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h] \in \underline{V}_T^h(\Gamma_{i,T}^h)$, $\vec{\rm m}_i^h \in \underline{V}_T^h(\gamma_T^h)$, $i = 1, 2$. Similarly, taking the time derivative of (4.13b) with $\partial_t^{\circ,h}\,\vec\chi = \vec 0$ yields, on noting (4.19), (4.21) and $\vec\kappa_\gamma^h \in \underline{V}_T^h(\gamma_T^h)$, that

$$\left\langle \partial_t^{\circ,h}\,\vec\kappa_\gamma^h, \vec\chi \right\rangle_{\gamma^h(t)}^h + \left\langle \vec\kappa_\gamma^h \,.\, \vec\chi, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)}^h + \left\langle \underline{\underline{\mathcal{P}}}_\gamma^h\,\vec\chi_s, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)} = 0\,. \tag{4.44}$$

Choosing $\vec\chi = \vec{\mathcal{V}}^h$ in (4.27a), $\vec\eta = \vec Y_i^h$ in (4.43), $i = 1, 2$, and combining yields, on noting the discrete variant of (3.16), that

$$\begin{aligned}
&\sum_{i=1}^2 \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h \\
&\quad + \sum_{i=1}^2 \Bigg[\tfrac12 \left\langle [\alpha_i\,|\vec\kappa_i^h - \overline\varkappa_i\,\vec\nu_i^h|^2 - 2\,\vec Y_i^h \,.\, \underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h]\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h \\
&\quad -\alpha_i\,\overline\varkappa_i \left\langle \vec\kappa_i^h, \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h), \vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle (\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h \,.\, \vec Y_i^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h \\
&\quad - \left\langle (1-\theta^h)\,(\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h) \,.\, \vec\nu_i^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h - \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h), \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h \Bigg] \\
&\quad + \varsigma \left\langle \vec{\rm id}_s, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)} - \sum_{i=1}^2 \alpha_i^G \left[\left\langle \vec\kappa_\gamma^h \,.\, \vec{\rm m}_i^h, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)}^h + \left\langle \underline{\underline{\mathcal{P}}}_\gamma^h\,(\vec{\rm m}_i^h)_s, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)} \right] \\
&= \sum_{i=1}^2 \left[\left\langle \partial_t^{\circ,h}\,\vec{\rm m}_i^h, \vec Y_i^h \right\rangle_{\gamma^h(t)} + \left\langle \vec{\rm m}_i^h \,.\, \vec Y_i^h, \vec{\rm id}_s, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)} \right].
\end{aligned} \tag{4.45}$$

Choosing $\vec\chi = \sum_{i=1}^2 \alpha_i^G\,\vec{\rm m}_i^h$ in (4.44) and recalling (4.27d,e) and (4.19), it follows from (4.45) that

$$\begin{aligned}
&\sum_{i=1}^2 \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h + \sum_{i=1}^2 \Bigg[\tfrac12\,\alpha_i \left\langle |\vec\kappa_i^h - \overline\varkappa_i\,\vec\nu_i^h|^2\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h \\
&\quad -\alpha_i\,\overline\varkappa_i \left\langle \vec\kappa_i^h, \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h), \vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h \\
&\quad - \left\langle (1-\theta^h)\,(\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h) \,.\, \vec\nu_i^h)\,\nabla_s\,\vec{\rm id}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h - \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h), \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h \Bigg] \\
&\quad + \varsigma \left\langle \vec{\rm id}_s, \vec{\mathcal{V}}_s^h \right\rangle_{\partial\Gamma^h(t)} = -\sum_{i=1}^2 \alpha_i^G \left[\left\langle \partial_t^{\circ,h}\,\vec{\rm m}_i^h, \vec\kappa_\gamma^h \right\rangle_{\gamma^h(t)}^h + \left\langle \vec{\rm m}_i^h, \partial_t^{\circ,h}\,\vec\kappa_\gamma^h \right\rangle_{\gamma^h(t)}^h + \left\langle \vec\kappa_\gamma^h \,.\, \vec{\rm m}_i^h, \vec{\rm id}_s \,.\, \vec{\mathcal{V}}_s^h \right\rangle_{\gamma^h(t)}^h \right] \\
&= -\frac{\rm d}{{\rm d}t} \sum_{i=1}^2 \alpha_i^G \left\langle \vec\kappa_\gamma^h, \vec{\rm m}_i^h \right\rangle_{\gamma^h(t)}^h.
\end{aligned} \tag{4.46}$$

We have from (4.10), (4.27f) and (4.4) that

$$\begin{aligned}
&\sum_{i=1}^2 \left[\left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h), \vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h - \alpha_i\,\overline\varkappa_i \left\langle \vec\kappa_i^h, \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h \right] \\
&\quad = \sum_{i=1}^2 \Bigg[\left\langle \partial_t^{\circ,h}\,\vec\kappa_i^h, \underline{\underline{Q}}_{i,\theta^h}^h\,\vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h - \alpha_i\,\overline\varkappa_i \left\langle \vec\kappa_i^h - \overline\varkappa_i\,\vec\nu_i^h, \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h \\
&\qquad\qquad + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h) - \underline{\underline{Q}}_{i,\theta^h}^h\,\partial_t^{\circ,h}\,\vec\kappa_i^h, \vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h \Bigg]
\end{aligned}$$

$$= \sum_{i=1}^{2} \left[\tfrac{1}{2}\,\alpha_i \left\langle \partial_t^{\circ,h}\, |\vec\kappa_i^h - \overline{\varkappa}_i\,\vec\nu_i^h|^2, 1 \right\rangle_{\Gamma_i^h(t)}^h + \left\langle \partial_t^{\circ,h}\,(\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h) - \underline{\underline{Q}}_{i,\theta^h}^h\,\partial_t^{\circ,h}\,\vec\kappa_i^h, \vec Y_i^h \right\rangle_{\Gamma_i^h(t)}^h \right]. \tag{4.47}$$

Combining (4.46) and (4.47), on noting (4.18), (4.19), (4.12), $\partial_t^{\circ,h}\,\theta^h = 0$ (which follows from (4.16) and (4.8)), $\vec\kappa_i^h \in \underline{V}_T^h(\Gamma_{i,T}^h)$, $i = 1, 2$, $\vec\nu_i^h\,|_{\sigma_{i,j}^h(\cdot)} \in \underline{V}_T^h(\sigma_{i,j,T}^h)$, $j = 1, \ldots, J_i$, $i = 1, 2$, (which follows from the discrete analogue of (3.16) and as $\vec{\mathrm{id}}\,|_{\Gamma^h(\cdot)} \in \underline{V}_T^h(\Gamma_T^h)$) and the invariance of $m(\Gamma_i^h(t))$ under continuous deformations, yields that

$$\sum_{i=1}^{2} \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h + \frac{\mathrm{d}}{\mathrm{d}t}\, E^h((\Gamma_i^h(t))_{i=1}^2) + \sum_{i=1}^{2} P_i = 0\,,$$

where, on noting (4.9),

$$P_i := \left\langle (1-\theta^h)\,\vec\kappa_i^h\,.\,\partial_t^{\circ,h}\,\vec\omega_i^h, \frac{\vec Y_i^h\,.\,\vec\omega_i^h}{|\vec\omega_i^h|^2} \right\rangle_{\Gamma_i^h(t)}^h + \left\langle (1-\theta^h)\,\vec Y_i^h\,.\,\partial_t^{\circ,h}\,\vec\omega_i^h, \frac{\vec\kappa_i^h\,.\,\vec\omega_i^h}{|\vec\omega_i^h|^2} \right\rangle_{\Gamma_i^h(t)}^h$$
$$- 2 \left\langle (1-\theta^h)\,(\vec\kappa_i^h\,.\,\vec\omega_i^h)\,(\vec Y_i^h\,.\,\vec\omega_i^h), \frac{\vec\omega_i^h\,.\,\partial_t^{\circ,h}\,\vec\omega_i^h}{|\vec\omega_i^h|^4} \right\rangle_{\Gamma_i^h(t)}^h - \left\langle (1-\theta^h)\,(\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h)\,.\,\vec\nu_i^h)\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h$$
$$- \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h), \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h, \quad i = 1, 2\,. \tag{4.48}$$

It remains to show that P_1 and P_2 as defined in (4.48) vanish. To see this, we observe that it follows from (4.26), (4.25) and the time derivative version of (4.23) that

$$P_i = \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h), \partial_t^{\circ,h}\,\vec\omega_i^h \right\rangle_{\Gamma_i^h(t)}^h + \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h)\,.\,(\vec\omega_i^h - \vec\nu_i^h)\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h$$
$$- \left\langle (1-\theta^h)\,\vec G_i^h(\vec Y_i^h, \vec\kappa_i^h), \partial_t^{\circ,h}\,\vec\nu_i^h \right\rangle_{\Gamma_i^h(t)}^h = 0\,, \quad i = 1, 2\,.$$

This proves the desired result (4.42).
Finally, if $\theta = 0$ then it immediately follows from (4.27b) that (4.41) holds. Hence $\Gamma_1^h(t)$ and $\Gamma_2^h(t)$ are open conformal polyhedral surfaces. ∎

Theorem 4.2. *Let* $\theta \in [0, 1]$, $\varrho \geq 0$ *and let* $\{(\Gamma^h, \vec\kappa_1^h, \vec\kappa_2^h, \vec Y_1^h, \vec Y_2^h, \vec\kappa_\gamma^h, \vec{\mathrm{m}}_1^h, \vec{\mathrm{m}}_2^h, \vec\Phi^h, \lambda^{V,h}, \lambda_1^{A,h}, \lambda_2^{A,h})(t)\}_{t\in[0,T]}$ *be a solution to* (4.32)*,* (4.27b–f) *and* (4.38a)*. In addition, we assume that* $\vec\kappa_\gamma^h \in \underline{V}_T^h(\gamma_T^h)$, $\vec\kappa_i^h, \vec\pi_i^h[\underline{\underline{Q}}_{i,\theta^h}^h\,\vec\kappa_i^h] \in \underline{V}_T^h(\Gamma_{i,T}^h)$, $\vec{\mathrm{m}}_i^h \in \underline{V}_T^h(\gamma_T^h)$, $i = 1, 2$*. Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, E^h((\Gamma_i^h(t))_{i=1}^2) = -\sum_{i=1}^{2} \left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h - \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h, \tag{4.49}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{H}^{d-1}(\Gamma_i^h(t)) = 0\,, \ i = 1, 2\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{L}^d(\Omega^h(t)) = 0\,, \tag{4.50}$$

where $\Omega^h(t)$ *denotes the region bounded by* $\Gamma^h(t)$*. Moreover, if* $\theta = 0$ *then* $\Gamma_1^h(t)$ *and* $\Gamma_2^h(t)$ *are open conformal polyhedral surfaces for all* $t \in (0, T]$*.*

Proof. We recall that on choosing $(\lambda^{V,h}, \lambda_1^{A,h}, \lambda_2^{A,h})$ solving the system (4.38a) yields that (4.29) and (4.30) hold, and hence the desired results (4.50) hold. The stability result (4.49) directly follows from the proof of Theorem 4.1. In particular, choosing $\vec\chi = \vec{\mathcal{V}}^h$ in (4.32), on noting (4.29) and (4.30), yields that

$$\left\langle \underline{\underline{Q}}_{i,\theta_\star^h}^h\,\vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma_i^h(t)}^h + \varrho \left\langle \vec{\mathcal{V}}^h, \vec{\mathcal{V}}^h \right\rangle_{\gamma^h(t)}^h = \left\langle \vec r^h, \vec{\mathcal{V}}^h \right\rangle_{\Gamma^h(t)}^h.$$

Combining this with (4.43) yields that (4.45) holds, and the rest of the proof proceeds as that of Theorem 4.1. Finally, as in the proof of Theorem 4.1, for $\theta = 0$ it follows from (4.27b) that $\Gamma^h_1(t)$ and $\Gamma^h_2(t)$ are conformal polyhedral surfaces. ■

5. **Fully discrete finite element approximation**

In this section we consider a fully discrete variant of the scheme (4.27a–f) from Section 4. To this end, let $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ be a partitioning of $[0, T]$ into possibly variable time steps $\Delta t_m := t_{m+1} - t_m$, $m = 0, \ldots, M-1$. Let Γ^m be a $(d-1)$-dimensional polyhedral surface, approximating $\Gamma^h(t_m)$, $m = 0, \ldots, M$, with the two parts Γ^m_i, $i = 1, 2$ and their common boundary γ^m. Following [19], we now parameterize the new surface Γ^{m+1} over Γ^m. Hence, we introduce the following finite element spaces. Let $\Gamma^m = \bigcup_{j=1}^J \overline{\sigma^m_j}$, where $\{\sigma^m_j\}_{j=1}^J$ is a family of mutually disjoint open triangles with vertices $\{\vec{q}^m_k\}_{k=1}^K$. Then for $m = 0, \ldots, M-1$, let

$$\underline{V}^h(\Gamma^m) := \{\vec{\chi} \in [C(\Gamma^m)]^d : \vec{\chi}|_{\sigma^m_j} \text{ is linear } \forall\, j = 1, \ldots, J\} =: [W^h(\Gamma^m)]^d . \tag{5.1}$$

We denote the standard basis of $W^h(\Gamma^m)$ by $\{\chi^m_k\}_{k=1}^K$. In addition, similarly to the semidiscrete setting in Section 4, we introduce the spaces $W^h(\Gamma^m_i)$ and $\underline{V}^h(\Gamma^m_i)$, denoting the standard basis of $W^h(\Gamma^m_i)$ by $\{\chi^m_{i,k}\}_{k=1}^{K_i}$, as well as $\underline{V}^h(\gamma^m)$, and the interpolation operators $\pi^m_i : C(\Gamma^m_i) \to W^h(\Gamma^m_i)$ and similarly $\vec{\pi}^m_i : [C(\Gamma^m_i)]^d \to \underline{V}^h(\Gamma^m_i)$.

We also introduce the L^2–inner products $\langle \cdot, \cdot \rangle_{\Gamma^m}$, $\langle \cdot, \cdot \rangle_{\Gamma^m_i}$ and $\langle \cdot, \cdot \rangle_{\gamma^m}$, as well as their mass lumped inner variants $\langle \cdot, \cdot \rangle^h_{\Gamma^m}$, $\langle \cdot, \cdot \rangle^h_{\Gamma^m_i}$ and $\langle \cdot, \cdot \rangle^h_{\gamma^m}$. Similarly to (4.3) and (4.5) we introduce the discrete vertex normals $\vec{\omega}^m_i := \sum_{k=1}^{K_i} \chi^m_{i,k}\, \vec{\omega}^m_{i,k} \in \underline{V}^h(\Gamma^m_i)$ and $\vec{\omega}^m := \sum_{k=1}^{K} \chi^m_k\, \vec{\omega}^m_k \in \underline{V}^h(\Gamma^m)$.

We make the following mild assumption.

($\mathcal{A}$) We assume for $m = 0, \ldots, M-1$ that $\mathcal{H}^{d-1}(\sigma^m_j) > 0$ for $j = 1, \ldots, J$, and that $\vec{0} \notin \{\vec{\omega}^m_{i,k} : k = 1, \ldots, K_i\,, i = 1, 2\}$. Moreover, in the case $C_1 = 1$ and $\theta = 0$ we assume that $\dim \operatorname{span}\{\vec{\omega}^m_{i,k} : k = 1, \ldots, K_i\,, i = 1, 2\} = d$, for $m = 0, \ldots, M-1$.

In addition, and similarly to (4.8) and (4.9), we introduce θ^m and $\theta^m_\star \in W^h(\Gamma^m)$, and then $\underline{\underline{Q}}^m_{i,\theta^m}$, $\underline{\underline{Q}}^m_{i,\theta^m_\star} \in [W^h(\Gamma^m_i)]^{d\times d}$, by setting $\underline{\underline{Q}}^m_{i,\theta^m}(\vec{q}^m_{i,k}) = \theta^m(\vec{q}^m_{i,k})\, \vec{\mathrm{Id}} + (1 - \theta^m(\vec{q}^m_{i,k}))\, |\vec{\omega}^m_{i,k}|^{-2}\, \vec{\omega}^m_{i,k} \otimes \vec{\omega}^m_{i,k}$ and $\underline{\underline{Q}}^m_{i,\theta^m_\star}(\vec{q}^m_{i,k}) = \theta^m_\star(\vec{q}^m_{i,k})\, \vec{\mathrm{Id}} + (1 - \theta^m_\star(\vec{q}^m_{i,k}))\, |\vec{\omega}^m_{i,k}|^{-2}\, \vec{\omega}^m_{i,k} \otimes \vec{\omega}^m_{i,k}$ for $k = 1, \ldots, K_i$, $i = 1, 2$. Similarly to (4.25) and (4.22), we let

$$\vec{G}^m_i(\vec{\xi}, \vec{\eta}) = \frac{1}{|\vec{\omega}^m_i|^2} \left((\vec{\xi} \,.\, \vec{\omega}^m_i)\, \vec{\eta} + (\vec{\eta} \,.\, \vec{\omega}^m_i)\, \vec{\xi} - 2\, \frac{(\vec{\eta} \,.\, \vec{\omega}^m_i)\, (\vec{\xi} \,.\, \vec{\omega}^m_i)}{|\vec{\omega}^m_i|^2}\, \vec{\omega}^m_i \right)$$

and

$$\underline{\underline{\mathcal{P}}}^m_\gamma = \underline{\underline{\mathrm{Id}}} - \vec{\mathrm{id}}_s \otimes \vec{\mathrm{id}}_s \quad \text{on} \quad \gamma^m .$$

On recalling (4.28), we consider the following fully discrete approximation of (4.27a–f). For $m = 0, \ldots, M-1$, find $\vec{X}^{m+1} \in \underline{V}^h(\Gamma^m)$, $(\vec{Y}^{m+1}_i, \vec{\mathrm{m}}^{m+1}_i)_{i=1}^2 \in \underline{V}^h(\Gamma^m_1) \times \underline{V}^h(\gamma^m) \times \underline{V}^h(\Gamma^m_2) \times \underline{V}^h(\gamma^m)$, $\vec{\kappa}^{m+1}_\gamma \in \underline{V}^h(\gamma^m)$ and $C_1\, \Phi^{m+1} \in \underline{V}^h(\gamma^m)$ such that

$$\sum_{i=1}^2 \left[\left\langle \underline{\underline{Q}}^m_{i,\theta^m_\star}\, \frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\chi} \right\rangle^h_{\Gamma^m_i} - \left\langle \nabla_s\, \vec{Y}^{m+1}_i, \nabla_s\, \vec{\chi} \right\rangle_{\Gamma^m_i} + \alpha^G_i \left\langle (\vec{\mathrm{m}}^{m+1}_i)_s, \vec{\chi}_s \right\rangle_{\gamma^m} \right]$$

$$
\begin{aligned}
&+ \varsigma \left\langle \vec{X}_s^{m+1}, \vec{\chi}_s \right\rangle_{\gamma^m} + \varrho \left\langle \frac{\vec{X}^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\chi} \right\rangle_{\gamma^m}^h \\
&= \sum_{i=1}^{2} \Big[\left\langle \nabla_s \,.\, \vec{Y}_i^m, \nabla_s \,.\, \vec{\chi} \right\rangle_{\Gamma_i^m} - \left\langle (\nabla_s \vec{Y}_i^m)^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s \vec{\mathrm{id}})^T \right\rangle_{\Gamma_i^m} \\
&\quad -\tfrac{1}{2} \left\langle [\alpha_i\, |\vec{\kappa}_i^m - \overline{\varkappa}_i\, \vec{\nu}_i^m|^2 - 2\,(\vec{Y}_i^m \,.\, \underline{\underline{Q}}_{i,\theta^m}^m\, \vec{\kappa}_i^m)]\, \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m}^h - \alpha_i\, \overline{\varkappa}_i \left\langle \vec{\kappa}_i^m, [\nabla_s \vec{\chi}]^T\, \vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \\
&\quad + \left\langle (1-\theta^m)\,(\vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m) \,.\, \vec{\nu}_i^m)\, \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m}^h - \left\langle (1-\theta^m)\, \vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m), [\nabla_s \vec{\chi}]^T\, \vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \Big] \\
&\quad + \sum_{i=1}^{2} \alpha_i^G \Big[\left\langle \vec{\kappa}_\gamma^m \,.\, \vec{\mathrm{m}}_i^m, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\gamma^m}^h + \left\langle (\underline{\underline{\mathrm{Id}}} + \underline{\underline{\mathcal{P}}}_\gamma^m)\, (\vec{\mathrm{m}}_i^m)_s, \vec{\chi}_s \right\rangle_{\gamma^m} \Big] \\
&\quad - \lambda^{V,m}\, \langle \vec{\omega}^m, \vec{\chi} \rangle_{\Gamma^m}^h - \sum_{i=1}^{2} \lambda_i^{A,m} \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m} \qquad \forall\ \vec{\chi} \in \underline{V}^h(\Gamma^m)\,, \qquad (5.2a)
\end{aligned}
$$

$$
\alpha_i^{-1} \left\langle \underline{\underline{Q}}_{i,\theta^m}^m\, \vec{Y}_i^{m+1}, \underline{\underline{Q}}_{i,\theta^m}^m\, \vec{\eta} \right\rangle_{\Gamma_i^m}^h + \left\langle \nabla_s \vec{X}^{m+1}, \nabla_s \vec{\eta} \right\rangle_{\Gamma_i^m} = \left\langle \vec{\mathrm{m}}_i^{m+1}, \vec{\eta} \right\rangle_{\gamma^m}^h - \overline{\varkappa}_i\, \langle \vec{\omega}_i^m, \vec{\eta} \rangle_{\Gamma_i^m}^h \quad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma_i^m)\,,\ i = 1,2\,, \qquad (5.2b)
$$

$$
\left\langle \vec{\kappa}_\gamma^{m+1}, \vec{\chi} \right\rangle_{\gamma^m}^h + \left\langle \vec{X}_s^{m+1}, \vec{\chi}_s \right\rangle_{\gamma^m} = 0 \qquad \forall\ \vec{\chi} \in \underline{V}^h(\gamma^m)\,, \qquad (5.2c)
$$

$$
C_1\, (\vec{\mathrm{m}}_1^{m+1} + \vec{\mathrm{m}}_2^{m+1}) = \vec{0} \qquad \text{on} \quad \gamma^m\,, \qquad (5.2d)
$$

$$
\alpha_i^G\, \vec{\kappa}_\gamma^{m+1} + \vec{Y}_i^{m+1} + C_1\, \vec{\Phi}^{m+1} = \vec{0} \qquad \text{on} \quad \gamma^m\,,\ i = 1,2\,, \qquad (5.2e)
$$

and set $\vec{\kappa}_i^{m+1} = \alpha_i^{-1}\, \vec{\pi}_i^m\, [\underline{\underline{Q}}_{i,\theta^m}^m\, \vec{Y}_i^{m+1}] + \overline{\varkappa}_i\, \vec{\omega}_i^m$ and $\Gamma_i^{m+1} = \vec{X}^{m+1}(\Gamma_i^m)$, $i = 1,2$. For $m \geq 1$ we note that here and throughout, as no confusion can arise, we denote by $\vec{\kappa}_i^m$ the function $\vec{z} \in \underline{V}^h(\Gamma_i^m)$, defined by $\vec{z}(\vec{q}_{i,k}^m) = \vec{\kappa}_i^m(\vec{q}_{i,k}^{m-1})$, $k = 1 \to K_i$, where $\vec{\kappa}_i^m \in \underline{V}(\Gamma_i^{m-1})$ is given, and similarly for e.g. $\vec{Y}_i^m$, $\vec{\mathrm{m}}_i^m$ and $\vec{\kappa}_\gamma^m$.

We note that if $C_1 = \alpha_1^G = \alpha_2^G = 0$ then the weak conormals $\vec{\mathrm{m}}_i^{m+1}$ play no role in the evolution. However, for surface area conservation they do play a role also in that case, see (5.3c) below. We also remark that the parameter $\varrho \geq 0$ has a stabilizing effect on the evolution of γ^m. In practice, this was particularly useful for simulations involving surface area preservation, and for C^0 experiments with Gaussian curvature energy contributions.

Of course, (5.2a–e) with $\lambda^{V,m} = \lambda_1^{A,m} = \lambda_2^{A,m} = 0$ corresponds to a fully discrete approximation of (4.27a–f), on recalling (4.28). For a fully discrete approximation of the volume and/or surface area preserving flow, on recalling (4.38a–c), we let $(\lambda^{V,m}, \lambda_1^{A,m}, \lambda_2^{A,m})$ be the solution of

$$
\begin{pmatrix} \sum_{i=1}^2 a_{i,\theta}^m(\vec{\omega}_i^m, \vec{\omega}_i^m) & a_{1,\theta}^m(\vec{\kappa}_1^m, \vec{\omega}_1^m) & a_{2,\theta}^m(\vec{\kappa}_2^m, \vec{\omega}_2^m) \\ a_{1,\theta}^m(\vec{\kappa}_1^m, \vec{\omega}_1^m) & a_{1,\theta}^m(\vec{\kappa}_1^m, \vec{\kappa}_1^m) & 0 \\ a_{2,\theta}^m(\vec{\kappa}_2^m, \vec{\omega}_2^m) & 0 & a_{2,\theta}^m(\vec{\kappa}_2^m, \vec{\kappa}_2^m) \end{pmatrix} \begin{pmatrix} -\lambda^{V,m} \\ \lambda_1^{A,m} \\ \lambda_2^{A,m} \end{pmatrix} = \begin{pmatrix} b_0^m \\ b_1^m \\ b_2^m \end{pmatrix}, \qquad (5.3a)
$$

where, on noting the fully discrete variant of (4.7),

$$
\begin{aligned}
b_0^m &= \sum_{i=1}^{2} \left[\left\langle (\vec{\Pi}_{i,0}^m - \underline{\underline{\mathrm{Id}}}) \frac{\vec{\mathrm{id}} - \vec{X}^{m-1}}{\Delta t_{m-1}}, \vec{\omega}^m \right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s \vec{Y}_i^m, \nabla_s\, (\vec{\Pi}_0^m\, \vec{\omega}^m) \right\rangle_{\Gamma_i^m} \right] - \left\langle \vec{f}^m, \vec{\Pi}_0^m\, \vec{\omega}^m \right\rangle_{\Gamma^m}^h \\
&= \sum_{i=1}^{2} \left[\left\langle (\vec{\Pi}_{i,0}^m - \underline{\underline{\mathrm{Id}}}) \frac{\vec{\mathrm{id}} - \vec{X}^{m-1}}{\Delta t_{m-1}}, \vec{\omega}_i^m \right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s \vec{Y}_i^m, \nabla_s\, (\vec{\Pi}_{i,0}^m\, \vec{\omega}_i^m) \right\rangle_{\Gamma_i^m} - \left\langle \vec{f}^m, \vec{\Pi}_{i,0}^m\, \vec{\omega}_i^m \right\rangle_{\Gamma_i^m}^h \right], \qquad (5.3b)
\end{aligned}
$$

$$b_i^m = \left\langle (\vec\Pi_{i,0}^m - \underline{\underline{\mathrm{Id}}})\,\frac{\vec{\mathrm{id}} - \vec X^{m-1}}{\Delta t_{m-1}}, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec\varkappa_i^m\right\rangle_{\Gamma_i^m}^h + \left\langle \vec{\mathrm{m}}_i^m, \frac{\vec{\mathrm{id}} - \vec X^{m-1}}{\Delta t_{m-1}}\right\rangle_{\gamma^m}^h$$
$$- \left\langle \nabla_s\,\vec Y_i^m, \nabla_s\,(\vec\Pi_{i,0}^m\,\vec\varkappa_i^m)\right\rangle_{\Gamma_i^m} - \left\langle \vec f^m, \vec\Pi_{i,0}^m\,\vec\varkappa_i^m\right\rangle_{\Gamma^m}^h$$
$$= \left\langle (\vec\Pi_{i,0}^m - \underline{\underline{\mathrm{Id}}})\,\frac{\vec{\mathrm{id}} - \vec X^{m-1}}{\Delta t_{m-1}}, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec\varkappa_i^m\right\rangle_{\Gamma_i^m}^h + \left\langle \vec{\mathrm{m}}_i^m, \frac{\vec{\mathrm{id}} - \vec X^{m-1}}{\Delta t_{m-1}}\right\rangle_{\gamma^m}^h$$
$$- \left\langle \nabla_s\,\vec Y_i^m, \nabla_s\,(\vec\Pi_{i,0}^m\,\vec\varkappa_i^m)\right\rangle_{\Gamma_i^m} - \left\langle \vec f^m, \vec\Pi_{i,0}^m\,\vec\varkappa_i^m\right\rangle_{\Gamma_i^m}^h, \quad i = 1, 2\,. \tag{5.3c}$$

Here, for convenience, we have re-written (5.2a) as

$$\sum_{i=1}^2 \left[\left\langle \underline{\underline{Q}}_{i,\theta^m_\star}^m\,\frac{\vec X^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec\chi\right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s\,\vec Y_i^{m+1}, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i^m} + \alpha_i^G\left\langle (\vec{\mathrm{m}}_i^{m+1})_s, \vec\chi_s\right\rangle_{\gamma^m}\right]$$
$$+ \varsigma\left\langle \vec X_s^{m+1}, \vec\chi_s\right\rangle_{\gamma^m} + \varrho\left\langle \frac{\vec X^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec\chi\right\rangle_{\gamma^m}^h$$
$$= \left\langle \vec f^m, \vec\chi\right\rangle_{\Gamma^m}^h - \lambda^{V,m}\,\langle \vec\omega^m, \vec\chi\rangle_{\Gamma^m}^h - \sum_{i=1}^2 \lambda_i^{A,m}\left\langle \nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i^m} \quad \forall\ \vec\chi \in \underline{V}^h(\Gamma^m)\,, \tag{5.4}$$

and, analogously to (4.34), we have defined $a_{i,\theta}^m : \underline{V}^h(\Gamma_i^m) \times \underline{V}^h(\Gamma_i^m) \to \mathbb{R}$ by setting

$$a_{i,\theta}^m(\vec\zeta, \vec\eta) = \left\langle \underline{\underline{Q}}_{i,\theta^m}^m\,\vec\zeta, \vec\Pi_{i,0}^m\,\vec\eta\right\rangle_{\Gamma_i^m}^h \qquad \forall\ \vec\zeta, \vec\eta \in \underline{V}^h(\Gamma_i^m)\,,\ i = 1, 2\,.$$

As before, we note that the matrix in (5.3a) is symmetric positive definite as long as $\vec\Pi_{i,0}^m\,\vec\omega_i^m$ and $\vec\Pi_{i,0}^m\,[\vec Q_{i,\theta^m}^m\,\vec\varkappa_i^m]$ are linearly independent, for $i = 1, 2$.

Theorem 5.1. *Let $\theta \in [0,1]$, $\varrho \geq 0$ and $\alpha_1, \alpha_2 > 0$. Let the assumptions $(\mathcal{A})$ hold. Then there exists a unique solution $\vec X^{m+1} \in \underline{V}^h(\Gamma^m)$, $(\vec Y_i^{m+1}, \vec{\mathrm{m}}_i^{m+1})_{i=1}^2 \in \underline{V}^h(\Gamma_1^m) \times \underline{V}^h(\gamma^m) \times \underline{V}^h(\Gamma_2^m) \times \underline{V}^h(\gamma^m)$, $\vec\varkappa_\gamma^{m+1} \in \underline{V}^h(\gamma^m)$ and $C_1\,\Phi^{m+1} \in \underline{V}^h(\gamma^m)$ to (5.2a–e).*

Proof. As (5.2a–e) is linear, existence follows from uniqueness. To investigate the latter, we consider the system: Find $\vec X \in \underline{V}^h(\Gamma^m)$, $(\vec Y_i, \vec{\mathrm{m}}_i)_{i=1}^2 \in \underline{V}^h(\Gamma_1^m) \times \underline{V}^h(\gamma^m) \times \underline{V}^h(\Gamma_2^m) \times \underline{V}^h(\gamma^m)$, $\vec\varkappa_\gamma \in \underline{V}^h(\gamma^m)$ and $C_1\,\Phi \in \underline{V}^h(\gamma^m)$ such that

$$\sum_{i=1}^2 \left[\frac{1}{\Delta t_m}\left\langle \underline{\underline{Q}}_{i,\theta^m_\star}^m\,\vec X, \vec\chi\right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s\,\vec Y_i, \nabla_s\,\vec\chi\right\rangle_{\Gamma_i^m} + \alpha_i^G\,\langle (\vec{\mathrm{m}}_i)_s, \vec\chi_s\rangle_{\gamma^m}\right] + \varsigma\left\langle \vec X_s, \vec\chi_s\right\rangle_{\gamma^m} + \varrho\left\langle \vec X, \vec\chi\right\rangle_{\gamma^m}^h = 0$$
$$\forall\ \vec\chi \in \underline{V}^h(\Gamma^m)\,, \tag{5.5a}$$

$$\alpha_i^{-1}\left\langle \underline{\underline{Q}}_{i,\theta^m}^m\,\vec Y_i, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec\eta\right\rangle_{\Gamma_i^m}^h + \left\langle \nabla_s\,\vec X, \nabla_s\,\vec\eta\right\rangle_{\Gamma_i^m} = \langle \vec{\mathrm{m}}_i, \vec\eta\rangle_{\gamma^m}^h \quad \forall\ \vec\eta \in \underline{V}^h(\Gamma_i^m)\,,\ i = 1, 2\,, \tag{5.5b}$$

$$\langle \vec\varkappa_\gamma, \vec\chi\rangle_{\gamma^m}^h + \left\langle \vec X_s, \vec\chi_s\right\rangle_{\gamma^m} = 0 \qquad \forall\ \vec\chi \in \underline{V}^h(\gamma^m)\,, \tag{5.5c}$$

$$C_1\,(\vec{\mathrm{m}}_1 + \vec{\mathrm{m}}_2) = \vec 0 \qquad \text{on} \quad \gamma^m\,, \tag{5.5d}$$

$$\alpha_i^G\,\vec\varkappa_\gamma + \vec Y_i + C_1\,\vec\Phi = \vec 0 \qquad \text{on} \quad \gamma^m\,,\ i = 1, 2\,. \tag{5.5e}$$

Choosing $\vec\chi = \vec X$ in (5.5a), $\vec\eta = \vec Y_i$ in (5.5b), $\vec\chi = \vec{\rm m}_i$ in (5.5c) and noting (5.5d,e), leads to

$$\sum_{i=1}^{2} \frac{1}{\Delta t_m} \left\langle \underline{\underline{Q}}^m_{i,\theta^m_\star}\, \vec X, \vec X\right\rangle^h_{\Gamma^m_i} + \varsigma \left\langle \vec X_s, \vec X_s\right\rangle_{\gamma^m} + \varrho \left\langle \vec X, \vec X\right\rangle^h_{\gamma^m}$$
$$= \sum_{i=1}^{2} \left[\left\langle \vec{\rm m}_i, \vec Y_i\right\rangle^h_{\gamma^m} - \alpha_i^{-1} \left\langle \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i, \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i\right\rangle^h_{\Gamma^m_i} - \alpha^G_i \left\langle (\vec{\rm m}_i)_s, \vec X_s\right\rangle_{\gamma^m}\right]$$
$$= \sum_{i=1}^{2} \left[\left\langle \vec Y_i, \vec{\rm m}_i\right\rangle^h_{\gamma^m} + \alpha^G_i \langle \vec{\rm m}_i, \vec\kappa_\gamma\rangle_{\gamma^m} - \alpha_i^{-1} \left\langle \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i, \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i\right\rangle^h_{\Gamma^m_i}\right]$$
$$= -\sum_{i=1}^{2} \alpha_i^{-1} \left\langle \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i, \underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i\right\rangle^h_{\Gamma^m_i} . \tag{5.6}$$

It follows from (5.6) and the definition of $\underline{\underline{Q}}^m_{i,\theta^m_\star}$ that $\vec X = \vec 0$ on γ^m, and so (5.5c) implies that $\vec\kappa_\gamma = \vec 0$. In addition, (5.6) yields that $\vec\pi^m_i\, [\underline{\underline{Q}}^m_{i,\theta^m}\, \vec Y_i] = \vec 0$, $i = 1, 2$. Hence, on adding the two equations in (5.5b) with $\vec\eta = \vec X$, we obtain that $\left\langle \nabla_s\, \vec X, \nabla_s\, \vec X\right\rangle_{\Gamma^m} = 0$, and so $\vec X = \vec 0$. Then (5.5b) implies that $\vec{\rm m}_1 = \vec{\rm m}_2 = \vec 0$. Next, we have from (5.5e), on recalling that $\vec\kappa_\gamma = \vec 0$, that there exists a $\vec Y \in \underline{V}^h(\Gamma^m)$ such that $\vec Y_i = \vec Y\,|_{\Gamma^m_i}$, $i = 1, 2$. Choosing $\vec\eta = \vec Y$ in (5.5a) yields that $\left\langle \nabla_s\, \vec Y, \nabla_s\, \vec Y\right\rangle_{\Gamma^m} = 0$, and hence $\vec Y$ is constant. If $C_1 = 0$ we immediately obtain from (5.5e) that $\vec Y = \vec 0$. If $C_1 = 1$, on the other hand, it follows that $\underline{\underline{Q}}^m_{i,\theta^m}(\vec q^m_{i,k})\, \vec Y = \vec 0$ for $k = 1, \ldots K_i$, $i = 1, 2$, and hence

$$\vec Y \,.\, \vec\omega^m_i(\vec q^m_{i,k}) = 0 \qquad k = 1, \ldots K_i,\ i = 1, 2. \tag{5.7}$$

The definition of $\underline{\underline{Q}}^m_{i,\theta^m}$, recall the fully discrete version of (4.9), and (5.7) then yield that $\theta\, \vec Y = \vec 0$. Hence for $\theta \in (0, 1]$ we immediately obtain that $\vec Y = \vec 0$, while in the case $\theta = 0$ it follows from assumption $(\mathcal{A})$ and (5.7) that $\vec Y = \vec 0$. Finally, we obtain that $\vec\Phi = \vec 0$ from (5.5e). ■

5.1. Implicit treatment of volume and area conservation

In practice it can be advantageous to consider implicit Lagrange multipliers $(\lambda^{V,m+1}, \lambda^{A,m+1}_1, \lambda^{A,m+1}_2)$ in order to obtain better discrete volume and surface area conservations. In particular, we replace (5.4) with

$$\sum_{i=1}^{2} \left[\left\langle \underline{\underline{Q}}^m_{i,\theta^m_\star} \frac{\vec X^{m+1} - \vec{\rm id}}{\Delta t_m}, \vec\chi\right\rangle^h_{\Gamma^m_i} - \left\langle \nabla_s\, \vec Y^{m+1}_i, \nabla_s\, \vec\chi\right\rangle_{\Gamma^m_i} + \alpha^G_i \left\langle (\vec{\rm m}^{m+1}_i)_s, \vec\chi_s\right\rangle_{\gamma^m}\right]$$
$$+ \varsigma \left\langle \vec X^{m+1}_s, \vec\chi_s\right\rangle_{\gamma^m} + \varrho \left\langle \frac{\vec X^{m+1} - \vec{\rm id}}{\Delta t_m}, \vec\chi\right\rangle^h_{\gamma^m}$$
$$= \left\langle \vec f^m, \vec\chi\right\rangle^h_{\Gamma^m} - \lambda^{V,m+1} \langle \vec\omega^m, \vec\chi\rangle^h_{\Gamma^m} - \sum_{i=1}^{2} \lambda^{A,m+1}_i \left\langle \nabla_s\, \vec X^{m+1}, \nabla_s\, \vec\chi\right\rangle_{\Gamma^m_i} \quad \forall\ \vec\chi \in \underline{V}^h(\Gamma^m)\,, \tag{5.8}$$

and require the coupled solutions $\vec X^{m+1} \in \underline{V}^h(\Gamma^m)$, $(\vec Y^{m+1}_i, \vec{\rm m}^{m+1}_i)_{i=1}^2 \in \underline{V}^h(\Gamma^m_1) \times \underline{V}^h(\gamma^m) \times \underline{V}^h(\Gamma^m_2) \times \underline{V}^h(\gamma^m)$, $\vec\kappa^{m+1}_\gamma \in \underline{V}^h(\gamma^m)$, $C_1\, \Phi^{m+1} \in \underline{V}^h(\gamma^m)$ and $(\lambda^{V,m+1}, \lambda^{A,m+1}_1, \lambda^{A,m+1}_2) \in \mathbb{R}^3$ to satisfy the nonlinear system (5.8), (5.2b–e) as well as an adapted variant of (5.3a–c), where the superscript m is replaced by $m+1$ in all occurrences of $\vec{\rm m}^m_i$, $\vec\kappa^m_i$, $\vec Y^m_i$, $\lambda^{V,m}$ and $\lambda^{A,m}_i$. In addition, $\frac{\vec{\rm id} - \vec X^{m-1}}{\Delta t_{m-1}}$ in (5.3b,c) is replaced by $\frac{\vec{X}^{m+1}-\vec{\mathrm{id}}}{\Delta t_m}$. In practice this nonlinear system can be solved with a fixed point iteration as follows. Let $(\lambda^{V,m+1,0}, \lambda_1^{A,m+1,0}, \lambda_2^{A,m+1,0}) = (\lambda^{V,m}, \lambda_1^{A,m}, \lambda_2^{A,m})$ and $\vec{X}^{m+1,0} = \vec{\mathrm{id}}\,|_{\Gamma^m}$. Then, for $j \geq 0$, find a solution $(\vec{X}^{m+1,j+1}, \vec{Y}^{m+1,j+1}, \vec{\kappa}_{\partial\Gamma}^{m+1,j+1}, \vec{\mathrm{m}}^{m+1,j+1})$ to the linear system (5.8), (5.2b–e), where any superscript $m+1$ on left hand sides is replaced by $m+1, j+1$, and by $m+1, j$ on the right hand side of (5.8). Then let $\vec{\kappa}_i^{m+1,j+1} = \alpha_i^{-1}\,\vec{\pi}_i^m\,[\underline{\underline{Q}}_{i,\theta^m}^m\,\vec{Y}_i^{m+1,j+1}] + \overline{\varkappa}_i\,\vec{\omega}_i^m$ be defined as usual, and compute $(\lambda^{V,m+1,j+1}, \lambda_1^{A,m+1,j+1}, \lambda_2^{A,m+1,j+1})$ as the unique solution to

$$\begin{pmatrix} \sum_{i=1}^2 a_{i,\theta}^m(\vec{\omega}_i^m, \vec{\omega}_i^m) & a_{1,\theta}^m(\vec{\kappa}_1^{m+1,j+1}, \vec{\omega}_1^m) & a_{2,\theta}^m(\vec{\kappa}_2^{m+1,j+1}, \vec{\omega}_2^m) \\ a_{1,\theta}^m(\vec{\kappa}_1^{m+1,j+1}, \vec{\omega}_1^m) & a_{1,\theta}^m(\vec{\kappa}_1^{m+1,j+1}, \vec{\kappa}_1^{m+1,j+1}) & 0 \\ a_{2,\theta}^m(\vec{\kappa}_2^{m+1,j+1}, \vec{\omega}_2^m) & 0 & a_{2,\theta}^m(\vec{\kappa}_2^{m+1,j+1}, \vec{\kappa}_2^{m+1,j+1}) \end{pmatrix} \begin{pmatrix} -\lambda^{V,m+1,j+1} \\ \lambda_1^{A,m+1,j+1} \\ \lambda_2^{A,m+1,j+1} \end{pmatrix} = \begin{pmatrix} b_0^{m+1,j+1} \\ b_1^{m+1,j+1} \\ b_2^{m+1,j+1} \end{pmatrix},$$

where

$$b_0^{m+1,j+1} = \sum_{i=1}^2 \left[\left\langle (\vec{\Pi}_{i,0}^m - \underline{\underline{\mathrm{Id}}})\,\frac{\vec{X}^{m+1,j+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\omega}_i^m \right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s\,\vec{Y}_i^{m+1,j+1}, \nabla_s\,(\vec{\Pi}_{i,0}^m\,\vec{\omega}_i^m) \right\rangle_{\Gamma_i^m} - \left\langle \vec{f}^m, \vec{\Pi}_{i,0}^m\,\vec{\omega}_i^m \right\rangle_{\Gamma_i^m}^h \right],$$

$$b_i^{m+1,j+1} = \left\langle (\vec{\Pi}_{i,0}^m - \underline{\underline{\mathrm{Id}}})\,\frac{\vec{X}^{m+1,j+1} - \vec{\mathrm{id}}}{\Delta t_m}, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec{\kappa}_i^{m+1,j+1} \right\rangle_{\Gamma_i^m}^h + \left\langle \vec{\mathrm{m}}_i^{m+1}, \frac{\vec{X}^{m+1,j+1} - \vec{\mathrm{id}}}{\Delta t_m} \right\rangle_{\gamma^m}^h - \left\langle \nabla_s\,\vec{Y}_i^{m+1,j+1}, \nabla_s\,(\vec{\Pi}_{i,0}^m\,\vec{\kappa}_i^{m+1,j+1}) \right\rangle_{\Gamma_i^m} - \left\langle \vec{f}^m, \vec{\Pi}_{i,0}^m\,\vec{\kappa}_i^{m+1,j+1} \right\rangle_{\Gamma_i^m}^h, \quad i = 1, 2\,;$$

and continue the iteration until

$$|\lambda^{V,m+1,j+1} - \lambda^{V,m+1,j}| + |\lambda_1^{A,m+1,j+1} - \lambda_1^{A,m+1,j}| + |\lambda_2^{A,m+1,j+1} - \lambda_2^{A,m+1,j}| < 10^{-8}\,.$$

We remark that the implicit scheme is chosen such that no new system matrices need to be assembled during the fixed point iteration. In particular, all integrals are evaluated on the old interfaces Γ_i^m. But all the quantities that are calculated during the linear solves are treated implicitly, i.e. $\vec{X}^{m+1}$, $(\vec{Y}_i^{m+1}, \vec{\kappa}_i^{m+1}, \vec{\mathrm{m}}_i^{m+1})_{i=1}^2$, $\vec{\kappa}_\gamma^{m+1}$, $C_1\,\Phi^{m+1}$, as well as the Lagrange multipliers.

6. **Solution methods**

Let us briefly outline how we solve the linear system (5.2a–e) in practice. First of all, similarly to our approach in [4] for the numerical approximation of surface clusters with triple junction lines, we reformulate (5.2a) as follows.

On introducing the following equivalent characterization of $\underline{V}^h(\Gamma^m)$, recall (5.1),

$$\underline{\hat{V}}^h(\Gamma^m) = \{(\vec{\eta}_1, \vec{\eta}_2) \in \mathop{\times}_{i=1}^{2} \underline{V}^h(\Gamma_i^m) : \vec{\eta}_1\,|_{\gamma^m} = \vec{\eta}_2\,|_{\gamma^m}\}\,,$$

we can rewrite (5.2a–e) equivalently as: Find $(\vec{X}_1^{m+1}, \vec{X}_2^{m+1}) \in \underline{\hat{V}}^h(\Gamma^m)$, $(\vec{Y}_i^{m+1}, \vec{\mathrm{m}}_i^{m+1})_{i=1}^2 \in \underline{V}^h(\Gamma_1^m) \times \underline{V}^h(\gamma^m) \times \underline{V}^h(\Gamma_2^m) \times \underline{V}^h(\gamma^m)$, $\vec{\kappa}_\gamma^{m+1} \in \underline{V}^h(\gamma^m)$ and $C_1\,\Phi^{m+1} \in \underline{V}^h(\gamma^m)$ such that

$$\sum_{i=1}^2 \left[\left\langle \underline{\underline{Q}}_{i,\theta_\star^m}^m\,\frac{\vec{X}_i^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\chi}_i \right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s\,\vec{Y}_i^{m+1}, \nabla_s\,\vec{\chi}_i \right\rangle_{\Gamma_i^m} + \alpha_i^G \left\langle (\vec{\mathrm{m}}_i^{m+1})_s, [\vec{\chi}_i]_s \right\rangle_{\gamma^m}\right.$$

$$
\begin{aligned}
&+\tfrac{1}{2}\,\varsigma \left\langle [\vec{X}_i^{m+1}]_s, [\vec{\chi}_i]_s \right\rangle_{\gamma^m} + \tfrac{1}{2}\,\varrho \left\langle \frac{\vec{X}_i^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\chi}_i \right\rangle_{\gamma^m}^h \Bigg] \\
&= \sum_{i=1}^{2} \Big[\left\langle \nabla_s \,.\, \vec{Y}_i^m, \nabla_s \,.\, \vec{\chi}_i \right\rangle_{\Gamma_i^m} - \left\langle (\nabla_s \vec{Y}_i^m)^T, \underline{\underline{D}}(\vec{\chi}_i)\,(\nabla_s \vec{\mathrm{id}})^T \right\rangle_{\Gamma_i^m} \\
&\quad -\tfrac{1}{2} \left\langle [\alpha_i\,|\vec{\kappa}_i^m - \overline{\varkappa}_i\,\vec{\nu}_i^m|^2 - 2\,(\vec{Y}_i^m \,.\, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec{\kappa}_i^m)]\,\nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi}_i \right\rangle_{\Gamma_i^m}^h - \alpha_i\,\overline{\varkappa}_i \left\langle \vec{\kappa}_i^m, [\nabla_s \vec{\chi}_i]^T\,\vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \\
&\quad + \left\langle (1-\theta^m)\,(\vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m)\,.\,\vec{\nu}_i^m)\,\nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi}_i \right\rangle_{\Gamma_i^m}^h - \left\langle (1-\theta^m)\,\vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m), [\nabla_s \vec{\chi}_i]^T\,\vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \Big] \\
&\quad + \sum_{i=1}^{2} \alpha_i^G \left[\left\langle \vec{\kappa}_\gamma^m \,.\, \vec{\mathrm{m}}_i^m, \vec{\mathrm{id}}_s \,.\, [\vec{\chi}_i]_s \right\rangle_{\gamma^m}^h + \left\langle (\underline{\underline{\mathrm{Id}}} + \underline{\underline{\mathcal{P}}}_\gamma^m)\,(\vec{\mathrm{m}}_i^m)_s, [\vec{\chi}_i]_s \right\rangle_{\gamma^m} \right] \\
&\quad - \lambda^{V,m} \sum_{i=1}^{2} \langle \vec{\omega}_i^m, \vec{\chi}_i \rangle_{\Gamma_i^m}^h - \sum_{i=1}^{2} \lambda_i^{A,m} \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi}_i \right\rangle_{\Gamma_i^m} \qquad \forall\ (\vec{\chi}_1, \vec{\chi}_2) \in \underline{\hat{V}}^h(\Gamma^m) \qquad (6.1)
\end{aligned}
$$

and (5.2b–e) hold, where in (6.1) we have used the fully discrete version of (4.7).

The above reformulation is crucial for the construction of fully practical solution methods, as it avoids the use of the global finite element space $\underline{V}^h(\Gamma^m)$. With the help of (6.1), it is now possible to work with the basis of the simple product finite element space $\underline{\hat{V}}^h(\Gamma^m)$, on employing suitable projections in the formulation of the linear problem. This construction is similar to e.g. the standard technique used for an ODE with periodic boundary conditions.

We recall from [10, (4.4a–d)] the following finite element approximation for Willmore flow of a single open surface $\Gamma_i(t)$ with free boundary conditions for $\partial\Gamma_i(t)$. For $m = 0, \ldots, M-1$, find $(\delta\vec{X}_i^{m+1}, \vec{Y}_i^{m+1}) \in \underline{V}^h(\Gamma_i^m) \times \underline{V}^h(\Gamma_i^m)$, with $\vec{X}_i^{m+1} = \vec{\mathrm{id}}\,|_{\Gamma_i^m} + \delta\vec{X}_i^{m+1}$, and $(\vec{\kappa}_{\partial\Gamma_i}^{m+1}, \vec{\mathrm{m}}_i^{m+1}) \in [\underline{V}^h(\partial\Gamma_i^m)]^2$ such that

$$
\begin{aligned}
&\left\langle \underline{\underline{Q}}_{i,\theta^m_\star}^m \frac{\vec{X}_i^{m+1} - \vec{\mathrm{id}}}{\Delta t_m}, \vec{\chi} \right\rangle_{\Gamma_i^m}^h - \left\langle \nabla_s \vec{Y}_i^{m+1}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m} + \varsigma \left\langle [\vec{X}_i^{m+1}]_s, \vec{\chi}_s \right\rangle_{\partial\Gamma_i^m} + \alpha_i^G \left\langle [\vec{\mathrm{m}}_i^{m+1}]_s, \vec{\chi}_s \right\rangle_{\partial\Gamma_i^m} \\
&= \left\langle \nabla_s \,.\, \vec{Y}_i^m, \nabla_s \,.\, \vec{\chi} \right\rangle_{\Gamma_i^m} - \left\langle (\nabla_s \vec{Y}_i^m)^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s \vec{\mathrm{id}})^T \right\rangle_{\Gamma_i^m} - \alpha_i\,\overline{\varkappa}_i \left\langle \vec{\kappa}_i^m, [\nabla_s \vec{\chi}]^T\,\vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \\
&\quad - \tfrac{1}{2} \left\langle \left[\alpha_i\,|\vec{\kappa}_i^m - \overline{\varkappa}_i\,\vec{\nu}_i^m|^2 - 2\,\vec{Y}_i^m \,.\, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec{\kappa}_i^m \right] \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m}^h \\
&\quad + \left\langle (1-\theta^m)\,(\vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m)\,.\,\vec{\nu}_i^m)\,\nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m}^h - \left\langle (1-\theta^m)\,\vec{G}_i^m(\vec{Y}_i^m, \vec{\kappa}_i^m), [\nabla_s \vec{\chi}]^T\,\vec{\nu}_i^m \right\rangle_{\Gamma_i^m}^h \\
&\quad + \alpha_i^G \left\langle \vec{\kappa}_{\partial\Gamma_i}^m \,.\, \vec{\mathrm{m}}_i^m, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\partial\Gamma_i^m}^h + \alpha_i^G \left\langle (\underline{\underline{\mathrm{Id}}} + \underline{\underline{\mathcal{P}}}_{\partial\Gamma_i}^m)\,[\vec{\mathrm{m}}_i^m]_s, \vec{\chi}_s \right\rangle_{\partial\Gamma_i^m} - \lambda_i^{A,m} \left\langle \nabla_s \vec{\mathrm{id}}, \nabla_s \vec{\chi} \right\rangle_{\Gamma_i^m} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ \vec{\chi} \in \underline{V}^h(\Gamma_i^m)\,, \qquad (6.2a)
\end{aligned}
$$

$$
\alpha_i^{-1} \left\langle \underline{\underline{Q}}_{i,\theta^m}^m\,\vec{Y}_i^{m+1}, \underline{\underline{Q}}_{i,\theta^m}^m\,\vec{\eta} \right\rangle_{\Gamma_i^m}^h + \left\langle \nabla_s \vec{X}_i^{m+1}, \nabla_s \vec{\eta} \right\rangle_{\Gamma_i^m} = \left\langle \vec{\mathrm{m}}_i^{m+1}, \vec{\eta} \right\rangle_{\partial\Gamma_i^m}^h - \overline{\varkappa}\,\langle \vec{\omega}_i^m, \vec{\eta} \rangle_{\Gamma_i^m}^h \quad \forall\ \vec{\eta} \in \underline{V}^h(\Gamma_i^m)\,, \qquad (6.2b)
$$

$$
\left\langle \alpha_i^G\,\vec{\kappa}_{\partial\Gamma_i}^{m+1} + \vec{Y}_i^{m+1}, \vec{\varphi} \right\rangle_{\partial\Gamma_i^m}^h = 0 \qquad \forall\ \vec{\varphi} \in \underline{V}^h(\partial\Gamma_i^m)\,, \qquad (6.2c)
$$

$$
\left\langle \vec{\kappa}_{\partial\Gamma_i}^{m+1}, \vec{\eta} \right\rangle_{\partial\Gamma_i^m}^h + \left\langle [\vec{X}_i^{m+1}]_s, \vec{\eta}_s \right\rangle_{\partial\Gamma_i^m} = 0 \quad \forall\ \vec{\eta} \in \underline{V}^h(\partial\Gamma_i^m)\,. \qquad (6.2d)
$$

The corresponding linear system from [10, (5.1)] is then given by

$$\begin{pmatrix} \vec{A} & -\frac{1}{\Delta t_m}\vec{\mathcal{M}}_{Q^\star} - \vec{A}_\varsigma & 0 & -\alpha_i^G\,\vec{A}_{\partial\Gamma,\Gamma} \\ \vec{\mathcal{M}}_{Q^2} & \vec{A} & 0 & -\vec{M}_{\partial\Gamma,\Gamma} \\ (\vec{M}_{\partial\Gamma,\Gamma})^T & 0 & \alpha_i^G\,\vec{M}_{\partial\Gamma} & 0 \\ 0 & (\vec{A}_{\partial\Gamma,\Gamma})^T & \vec{M}_{\partial\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \vec{Y}_i^{m+1} \\ \delta\vec{X}_i^{m+1} \\ \vec{\kappa}_{\partial\Gamma_i}^{m+1} \\ \vec{\mathrm{m}}_i^{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} [\vec{\mathcal{B}}^\star - \vec{\mathcal{B}} + \vec{\mathcal{R}}]\,\vec{Y}_i^m + (\vec{A}_\theta + \vec{A}_\varsigma + \lambda_i^{A,m}\,\vec{A})\,\vec{X}_i^m + \vec{b}_\theta - \vec{b}_\alpha \\ -\vec{A}\,\vec{X}_i^m - \overline{\varkappa}\,\vec{M}\,\vec{\omega}_i^m \\ \vec{0} \\ -(\vec{A}_{\partial\Gamma,\Gamma})^T\,\vec{X}_i^m \end{pmatrix}. \tag{6.3}$$

On replacing $\vec{A}_\varsigma$ with $(\frac{1}{2}\,\vec{A}_\varsigma + \frac{1}{2\Delta t}\,\vec{M}_\varrho)$, where the definition of $\vec{M}_\varrho$ is clear from (6.1), and similarly adapting the first entry in the right hand side of (6.3) to account for the term involving $\lambda^{V,m}$, we write (6.3) as

$$B_i\,Z_i = g_i\,.$$

Hence we can rewrite the linear system for (6.1), (5.2b–e) as

$$\mathcal{P}_B\,\mathcal{B}\,\mathcal{P}_Z \begin{pmatrix} Z_1 \\ Z_2 \\ \vec{\Phi}^{m+1} \end{pmatrix} = \mathcal{P}_B \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix}, \tag{6.4a}$$

where

$$\mathcal{B} = \begin{pmatrix} B_1 & 0 & \begin{pmatrix} 0 \\ 0 \\ C_1\,\vec{M}^\gamma \\ 0 \end{pmatrix} \\ 0 & B_2 & \begin{pmatrix} 0 \\ 0 \\ C_1\,\vec{M}^\gamma \\ 0 \end{pmatrix} \\ (0\;0\;0\;C_1\,\vec{M}^\gamma) & (0\;0\;0\;C_1\,\vec{M}^\gamma) & 0 \end{pmatrix}, \tag{6.4b}$$

and where $\vec{M}^\gamma$ is a mass matrix on γ^m. Moveover, $\mathcal{P}_B$ and $\mathcal{P}_Z$ are the orthogonal projections that encode the test and trial space $\underline{\hat{V}}^h(\Gamma^m)$ in (6.1), i.e. they act on the first and fifth block row in (6.4b), and on the second entries of Z_1 and Z_2, respectively.

The system (6.4a) can be efficiently solved in practice with a preconditioned BiCGSTAB or GMRES iterative solver, where we employ the preconditioners

$$\mathcal{P}_Z \begin{pmatrix} B_1^{-1} & 0 & 0 \\ 0 & B_2^{-1} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} \mathcal{P}_B \quad \text{and} \quad \mathcal{P}_Z\,\mathcal{B}^{-1}\,\mathcal{P}_B$$

for the cases $C_1 = 0$ and $C_1 = 1$, respectively. Here we recall from [10] that B_1 and B_2 are invertible. The inverses B_1^{-1} and B_2^{-1} can be computed with the help of the sparse factorization package UMFPACK, see [16]. Similarly, the inverse $\mathcal{B}^{-1}$, which existed in all our numerical tests, can also be computed with the help of UMFPACK.

In practice we note that the preconditioned Krylov subspace solvers usually take fewer than ten iterations per time step to converge. We stress that the chosen preconditioners are crucial, as without appropriate preconditioning the iterative solvers do not converge. This suggests that the linear systems (6.4a) are badly conditioned.

7. Numerical results

We implemented our fully discrete finite element approximations within the finite element toolbox ALBERTA, see [34]. The arising systems of linear equations were solved with the help of the sparse factorization package UMFPACK, see [16]. For the computations involving surface area preserving Willmore flow, we always employ the implicit Lagrange multiplier formulation discussed in §5.1.

The fully discrete scheme (5.2a–e) needs initial data $\vec\kappa_i^0$, $\vec Y_i^0$, $\vec{\rm m}_i^0$, $i = 1, 2$, and $\vec\kappa_\gamma^0$. Given the initial triangulation Γ_i^0, we let $\vec{\rm m}_i^0 \in \underline{V}^h(\gamma^0)$ be such that

$$\left\langle \vec{\rm m}_i^0, \vec\eta \right\rangle_{\gamma^0}^h = \left\langle \vec\mu_i^0, \vec\eta \right\rangle_{\gamma^0} \qquad \forall\ \vec\eta \in \underline{V}^h(\gamma^0)\,,$$

with $\vec\mu_i^0$ denoting the conormal on $\partial\Gamma_i^0$, $i = 1, 2$. In addition, we let

$$\vec\kappa_i^0 = -\tfrac{2}{R}\,\vec\omega_i^0$$

for simulations where $\Gamma_i(0)$ is part of a sphere of radius R, i.e. $\Gamma_i(0) \subset \partial B_R(\vec 0)$, and otherwise define $\vec\kappa_i^0 \in \underline{V}^h(\Gamma_i^0)$ to be the solution of

$$\left\langle \vec\kappa_i^0, \vec\eta \right\rangle_{\Gamma_i^0}^h + \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\eta \right\rangle_{\Gamma_i^0} = \left\langle \vec{\rm m}_i^0, \vec\eta \right\rangle_{\gamma^0}^h \qquad \forall\ \vec\eta \in \underline{V}^h(\Gamma_i^0)\,.$$

Then we define

$$\vec Y_i^0 = \alpha_i\,[\vec\kappa_i^0 - \overline{\varkappa}_i\,\vec\omega_i^0]\,.$$

Moreover, we let $\vec\kappa_\gamma^0 \in \underline{V}^h(\gamma^0)$ be such that

$$\left\langle \vec\kappa_\gamma^0, \vec\eta \right\rangle_{\gamma^0}^h + \left\langle \vec{\rm id}_s, \vec\eta_s \right\rangle_{\gamma^0} = 0 \quad \forall\ \vec\eta \in \underline{V}^h(\gamma^0)\,.$$

Throughout this section we use uniform time steps $\Delta t_m = \Delta t$, $m = 0, \ldots, M-1$, and set $\Delta t = 10^{-3}$ unless stated otherwise. In addition, unless stated otherwise, we fix $\alpha_i = 1$ and $\overline{\varkappa}_i = \alpha_i^G = 0$, $i = 1, 2$, as well as $\varsigma = 0$. At times we will discuss the discrete energy of the numerical solutions, which, similarly to (4.12), is defined by

$$E^{m+1}((\Gamma_i^m)_{i=1}^2) := \sum_{i=1}^2 \left[\tfrac12\,\alpha_i \left\langle |\vec\kappa_i^{m+1} - \overline{\varkappa}_i\,\vec\nu_i^m|^2, 1 \right\rangle_{\Gamma_i^m}^h + \alpha_i^G \left[\left\langle \vec\kappa_\gamma^{m+1}, \vec{\rm m}_i^{m+1} \right\rangle_{\gamma^m}^h + 2\,\pi\, m(\Gamma_i^m)\right]\right] + \varsigma\,\mathcal{H}^{d-2}(\gamma^m)\,.$$

Finally, we fix $\theta = 0$ throughout, unless otherwise stated.

For the visualization of our numerical results we will use the colour red for Γ_1^m, and the colour yellow for Γ_2^m.

7.1. The C^0–case

In Figure 7.1 we show the evolution of the outer shell of a torus joined with two spherical caps. Here the two caps make up phase 1, with the remainder representing phase 2. The initial surface Γ^0 satisfies $(J_1, J_2) = (2048, 4096)$ and $(K_1, K_2) = (1090, 2112)$ and has maximal dimensions $6 \times 6 \times 6$, i.e. up to translations, the smallest cuboid containing Γ^0 is $[0, 6]^3$. For the parameters $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$ and $\varsigma = 0.1$, the surface evolves towards a catenoid. In Figure 7.2 we show the same evolution for the values $\overline{\varkappa}_1 = -2$ and $\overline{\varkappa}_2 = -0.5$, which is now markedly different. The same evolution with $\varrho = 2$, which shows the slowing influence of $\varrho > 0$, is shown in Figure 7.3. In both experiments the effect of the two different spontaneous curvature values for the two phases can clearly be seen. The same evolution as in Figure 7.3, but now for surface area preserving flow, is shown in Figure 7.4. Here the observed relative surface area loss is 0.12%. The interplay between the different values of $\overline{\varkappa}_i$, the surface area

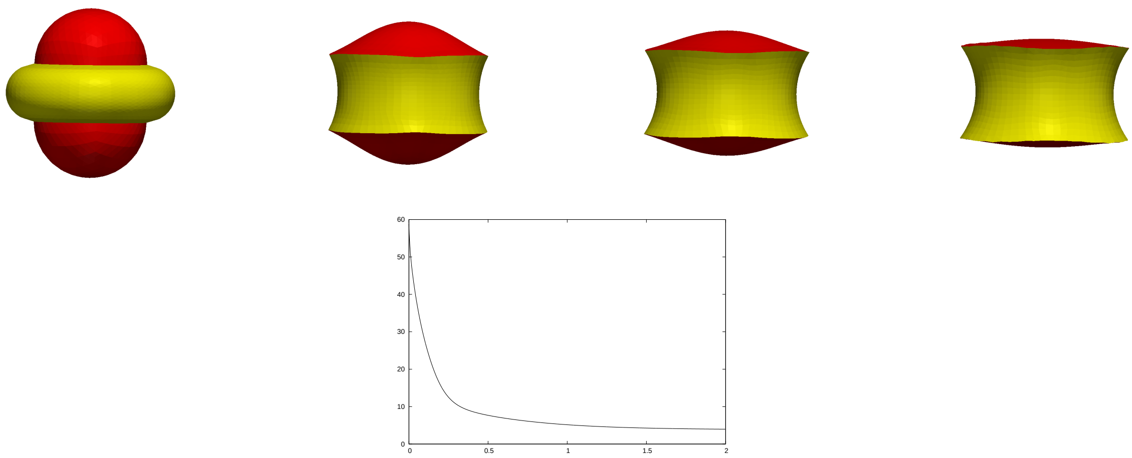

Figure 7.1. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 0.1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

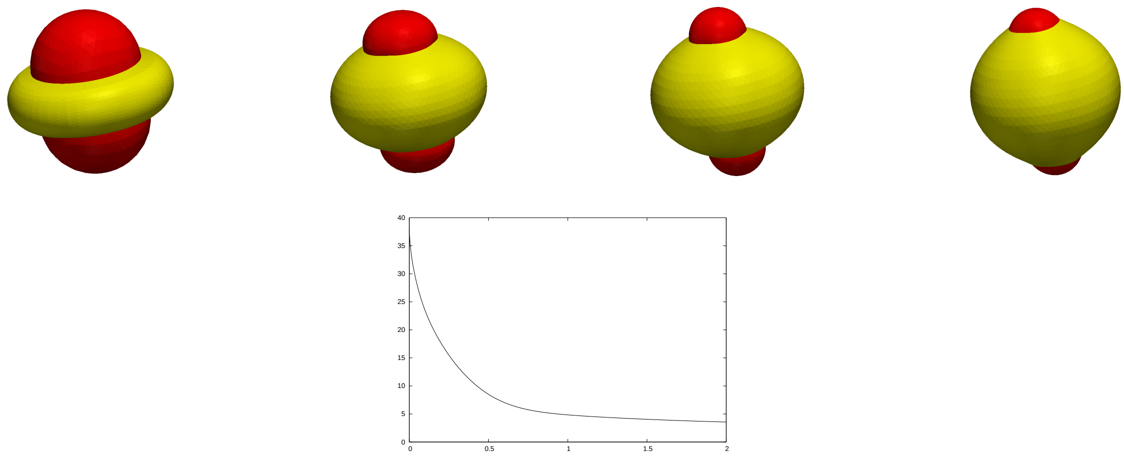

Figure 7.2. (C^0: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

constraints, and the C^0–attachment condition lead to an interesting evolution. A completely different evolution is obtained when we replace surface area conservation with volume conservation. This new simulation is visualized in Figure 7.5, where the observed relative volume loss is 0.00%.

A simulation with four disconnected components for phase 1 is shown in Figure 7.6. The initial surface Γ^0 satisfies $(J_1, J_2) = (1816, 4328)$ and $(K_1, K_2) = (1000, 2250)$ and has maximal dimensions $4.2 \times 4.2 \times 1.1$. The evolution for the parameters $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$ and $\varsigma = 1$ goes towards a fournoid.

We now consider surface area preserving experiments for setups where phase 1 is represented by six or eight disconnected components on the unit sphere. For these experiments we use the time step size $\Delta t = 10^{-4}$ and let $\overline{\varkappa}_1 = -4$, $\overline{\varkappa}_2 = -2$, $\varsigma = 1$ and $\varrho = 2$. The initial surface Γ^0 in Figure 7.7 satisfies $(J_1, J_2) = (1032, 7160)$ and $(K_1, K_2) = (614, 3668)$ and is an approximation of the unit sphere. Phase 1 is made up of six disconnected components. Here the observed relative surface area loss is 0.36%. A

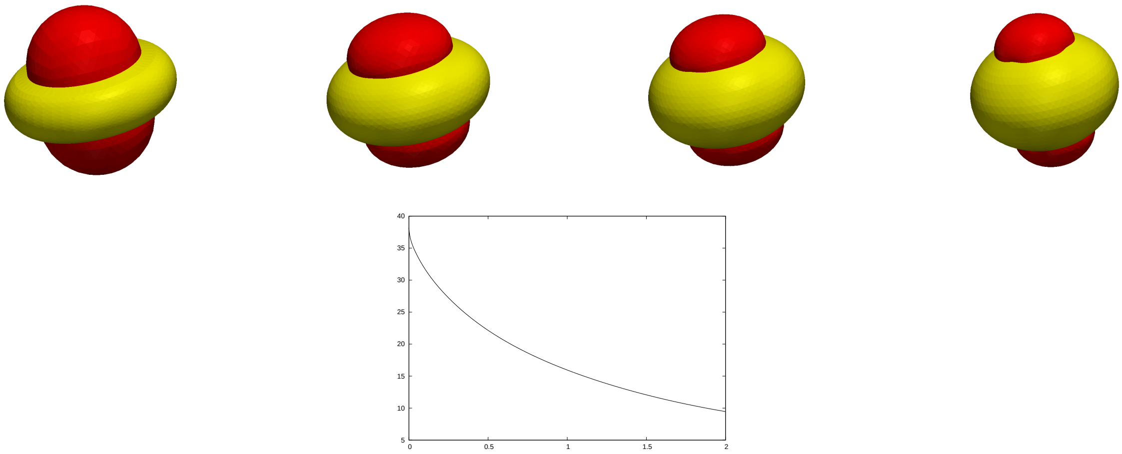

Figure 7.3. (C^0: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

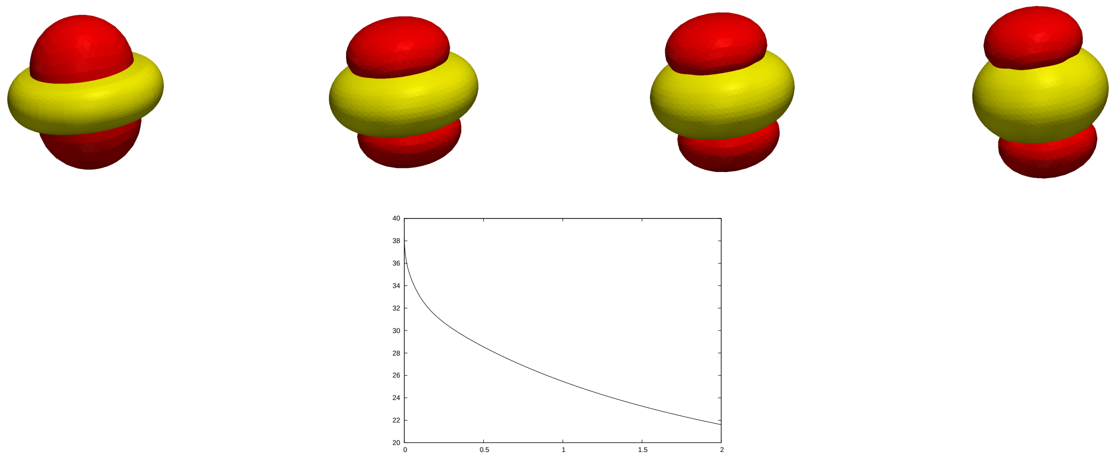

Figure 7.4. (C^0: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$) Surface area preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

simulation with eight disconnected components for phase 1 is shown in Figure 7.8. The initial surface Γ^0 satisfies $(J_1, J_2) = (2048, 6144)$ and $(K_1, K_2) = (1184, 3218)$. Here the observed relative surface area loss is 0.28%.

An example for volume and surface area preserving flow is shown in Figure 7.9. The initial surface Γ^0 satisfies $(J_1, J_2) = (2274, 2274)$ and $(K_1, K_2) = (1188, 1188)$ and has maximal dimensions $1.5\times1.5\times2.8$. In this experiment we choose $\overline{\varkappa}_1 = \overline{\varkappa}_2 = -1$, $\varsigma = 1$ and $\varrho = 2$. The relative surface area loss for this experiment is 0.07%, while the relative volume loss is 0.00%.

The next set of experiments illustrates the impact of the Gaussian curvature energy. The initial surface Γ^0 is made up of two halves of an approximation of the unit sphere and satisfies $(J_1, J_2) = (2274, 2274)$ and $(K_1, K_2) = (1188, 1188)$. An experiment for $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$ and $\varrho = 2$ is shown in Figure 7.10. The evolution eventually reaches a slowly shrinking disk. Choosing the parameters

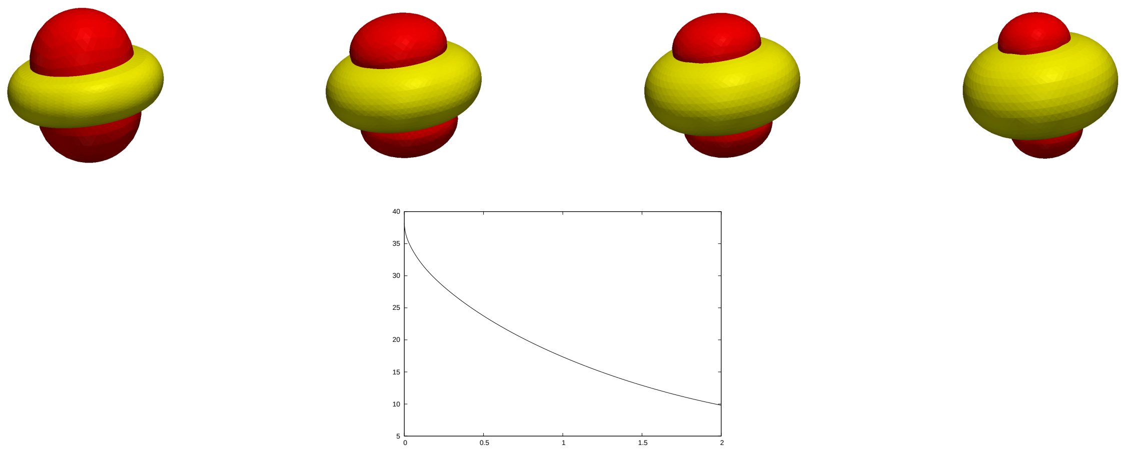

Figure 7.5. (C^0: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$) Volume preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.5,\ 1,\ 2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

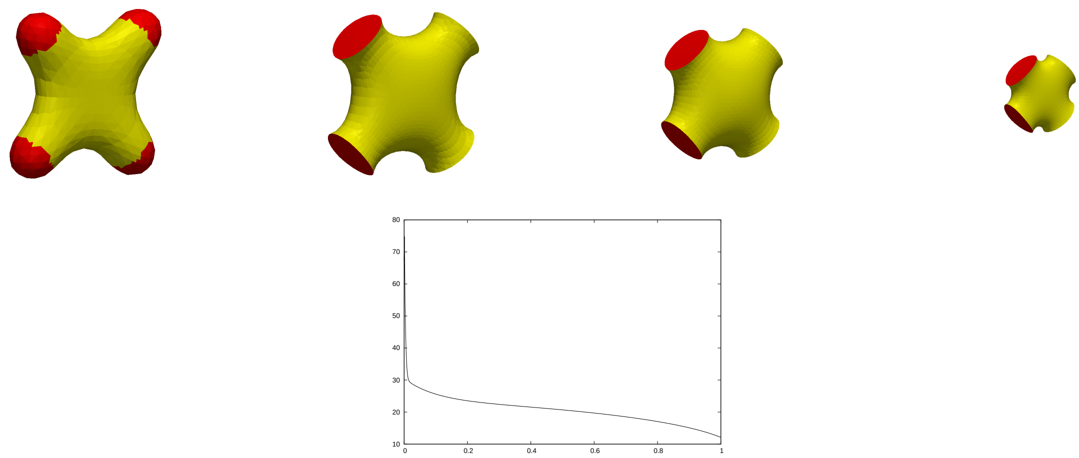

Figure 7.6. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.5,\ 1$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

$\alpha_1^G = \alpha_2^G = -1$, and using the time step size $\Delta t = 10^{-5}$, we obtain the simulation in Figure 7.11. We remark that the conditions (2.7) trivially hold. Moreover, and in contrast to the C^1–case, a nonzero Gaussian bending energy coefficient has an influence on the evolution even if $\alpha_1^G = \alpha_2^G$. In this example we observe that for a negative $\alpha_1^G = \alpha_2^G$, the term $\sum_{i=1}^2 \alpha_i^G \int_{\Gamma_i} \mathcal{K}_i \,\mathrm{d}\mathcal{H}^2$ for the initial sphere is negative, and hence the evolution remains convex throughout, in contrast to the evolution in Figure 7.10. Moreover, the evolution in Figure 7.11 is generally slower, since large values of the Gaussian curvatures make $\sum_{i=1}^2 \alpha_i^G \int_{\Gamma_i} \mathcal{K}_i \,\mathrm{d}\mathcal{H}^2$ more negative. Repeating the computation for $\alpha_1^G = -1$ and $\alpha_2^G = -1.5$ yields the results in Figure 7.12. We note once again that the conditions (2.7) hold. For the evolution in Figure 7.12 we observe that the curvature of phase 2 is decreasing slower due to the fact that large values of $\int_{\Gamma_2} \mathcal{K}_2 \,\mathrm{d}\mathcal{H}^2$ decrease the energy.

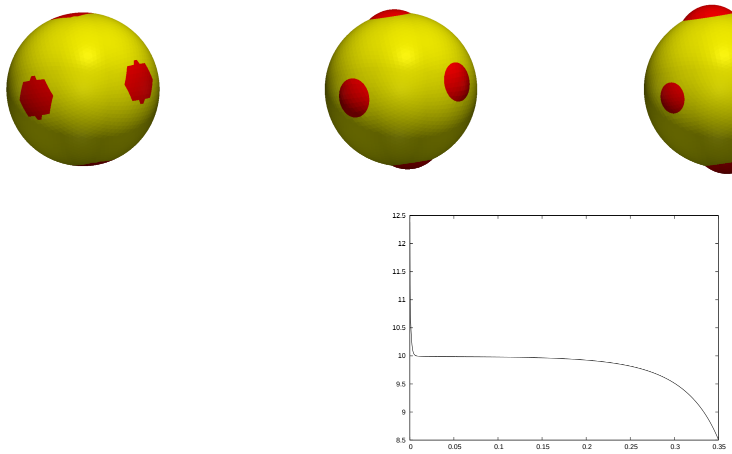

Figure 7.7. (C^0: $\overline{\varkappa}_1 = -4$, $\overline{\varkappa}_2 = -2$, $\varsigma = 1$, $\varrho = 2$) Surface area preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.3,\ 0.35$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

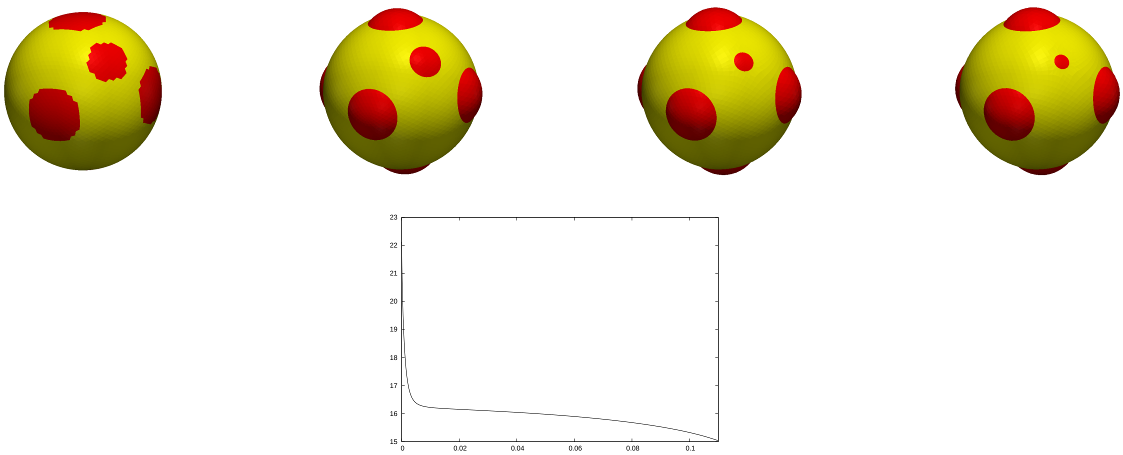

Figure 7.8. (C^0: $\overline{\varkappa}_1 = -4$, $\overline{\varkappa}_2 = -2$, $\varsigma = 1$, $\varrho = 2$) Surface area preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.05,\ 0.1,\ 0.11$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

7.2. The C^1–case

We remark that in the C^1–case, with uniform data $\alpha_1 = \alpha_2 = \alpha$, $\overline{\varkappa}_1 = \overline{\varkappa}_2 = \overline{\varkappa}$ and $\varsigma = \varrho = \alpha_1^G = \alpha_2^G = 0$, our finite element approximation collapses to the scheme from [8] for the Willmore flow of closed surfaces. Indeed, as a numerical check we confirmed that Table 1 in [8], for the approximation of the nonlinear ODE [8, (5.1)], is reproduced exactly by our implementation of the scheme (5.2a–e).

A repeat of the simulation in Figure 7.3 in the context of a C^1–condition on γ is shown in Figure 7.13, where for this experiment we use the time step size $\Delta t = 10^{-4}$. The evolution goes towards a cylinder with two round caps, which is dramatically different to the evolution in the C^0–case.

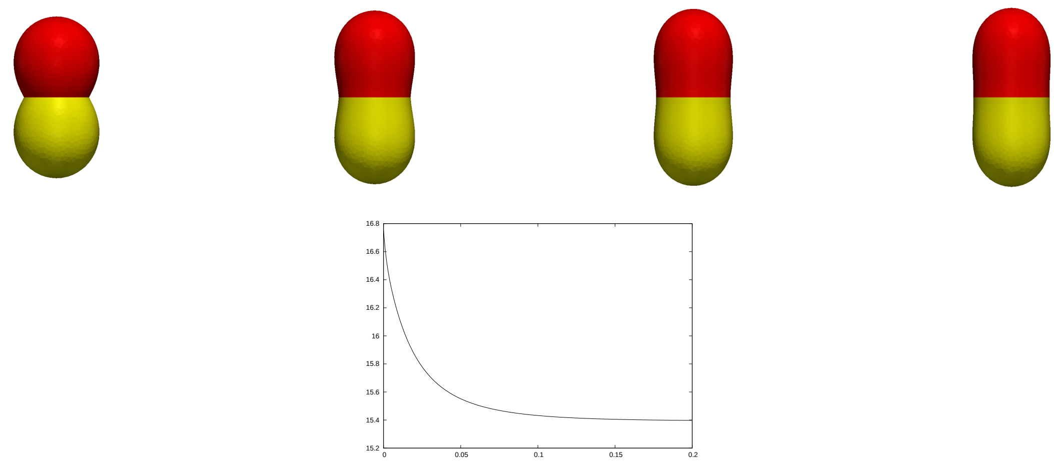

Figure 7.9. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = -1$, $\varsigma = 1$, $\varrho = 2$) Volume and surface area preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.05,\ 0.1,\ 0.2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

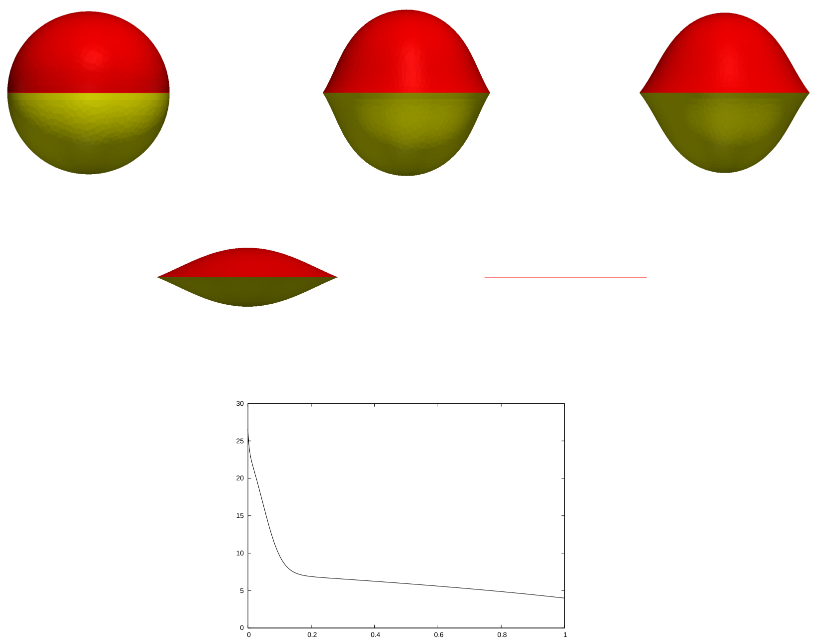

Figure 7.10. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$, $\varrho = 2$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.01,\ 0.02,\ 0.1,\ 1$. At time $t = 1$ the evolution has reached a disk. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

If we project the initial surface from Figure 7.13 to the unit sphere, we obtain the evolution shown in Figure 7.14. The evolution goes towards a cylinder with two round caps. Using the same parameters as in Figure 7.14 to simulate surface area preserving flow, we obtain the evolution shown in Figure 7.15, where here we have chosen $\varrho = 2$. The evolution goes towards a more elongated cylinder with two

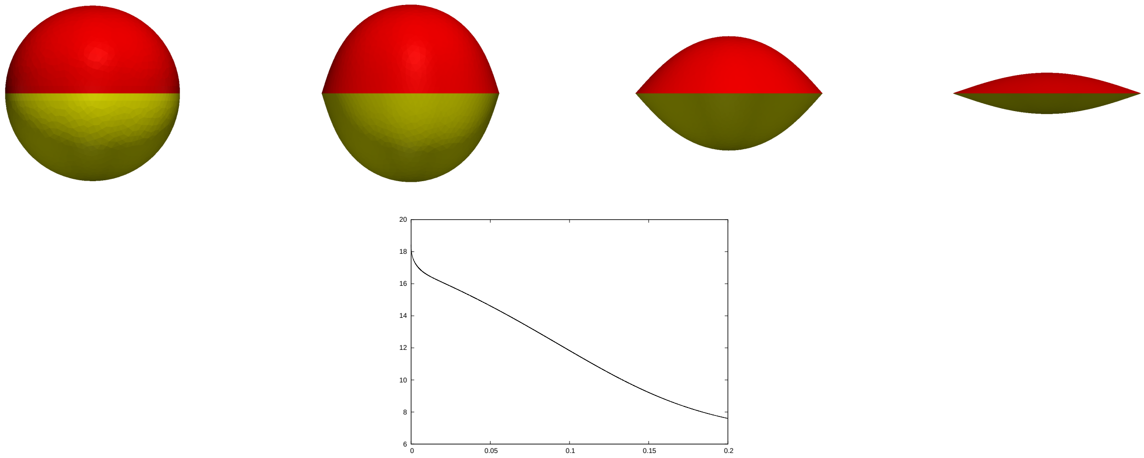

FIGURE 7.11. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$, $\alpha_1^G = \alpha_2^G = -1$, $\varrho = 2$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.01,\ 0.1,\ 0.2$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

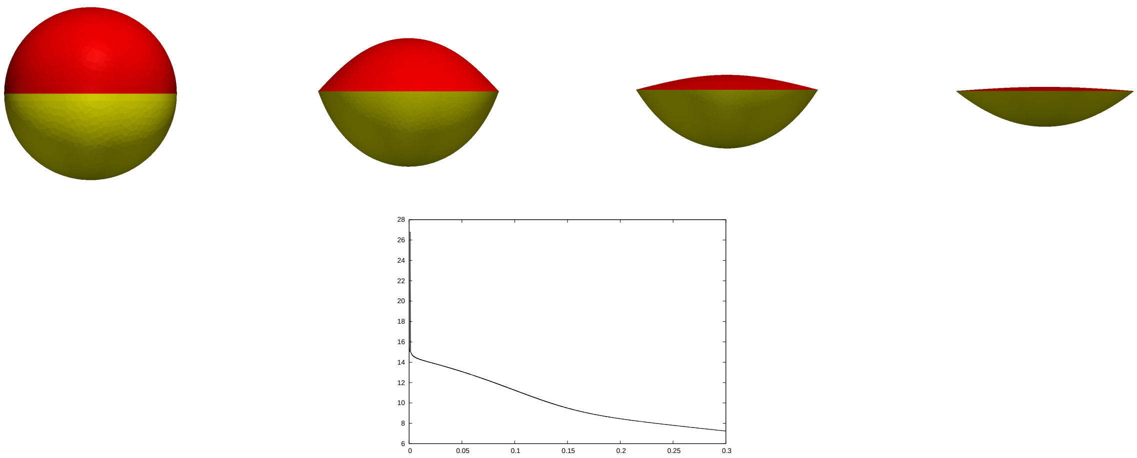

FIGURE 7.12. (C^0: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$, $\alpha_1^G = -1$, $\alpha_2^G = -1.5$, $\varrho = 2$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.2,\ 0.3$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

round caps. Here the observed relative surface area loss is 0.18%. The volume preserving variant is shown in Figure 7.16, where in order to dampen the tangential motion we choose $\theta = 0.05$. Here the observed relative volume loss is -0.12%.

A repeat of the simulation in Figure 7.6, now in the context of a C^1–condition on γ, is shown in Figure 7.17. Once again, we observe that the C^1–condition has a dramatic effect on the evolution.

In the next experiments we investigate the possible influence of the Gaussian curvature energy. If we choose the initial surface as in Figure 7.11, and running with $\overline{\varkappa}_1 = \overline{\varkappa}_2 = -0.5$ and $\varsigma = \alpha_1^G = \alpha_2^G = 0$, then we obtain an expanding sphere, with symmetric phases 1 and 2, which approximates the solution to the nonlinear ODE [8, (5.1)]. On the continuous level, thanks to the Gauss–Bonnet theorem, the same solution is obtained when choosing $\alpha_1^G = \alpha_2^G = 0.5$, and this is also replicated by our numerical approximation. Choosing $\alpha_1^G = 0.5$ and $\alpha_2^G = 1$, on the other hand, leads to phase 1 growing on the

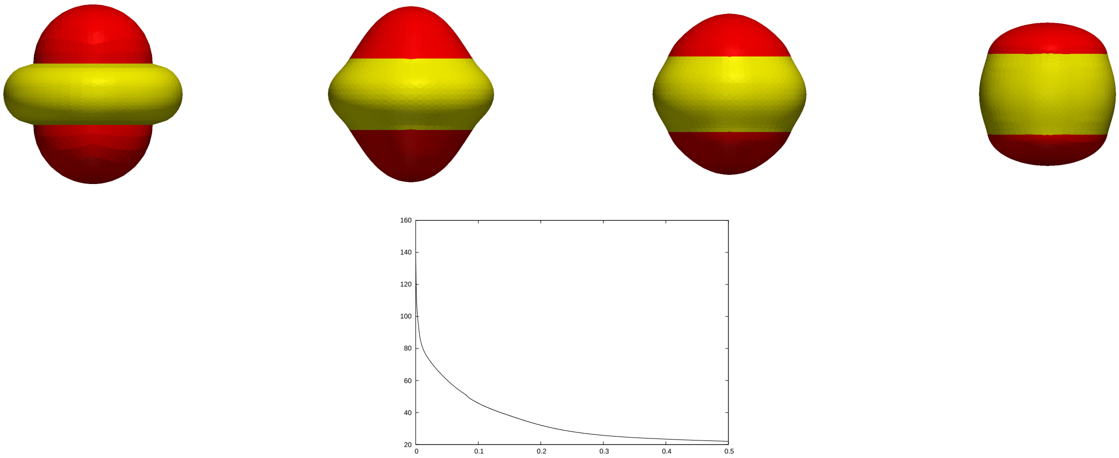

Figure 7.13. (C^1: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.2,\ 0.5$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

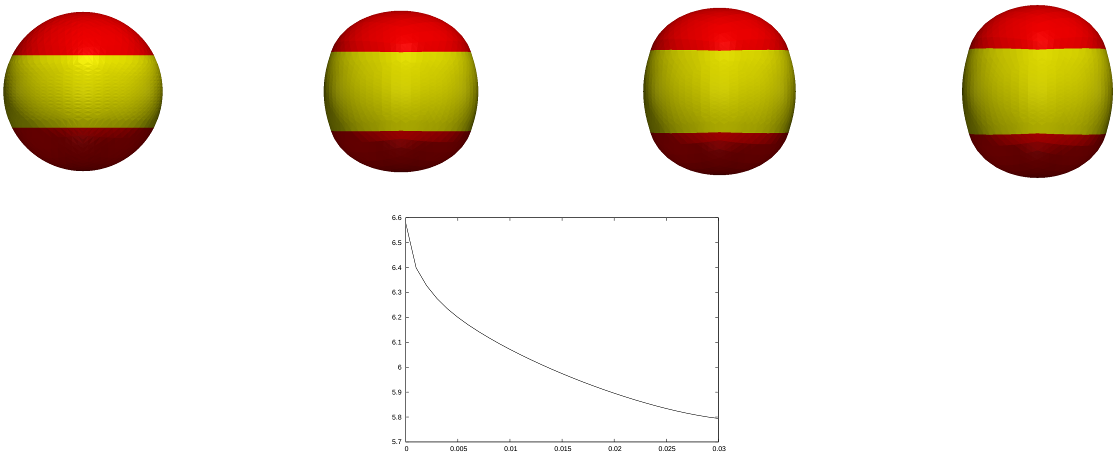

Figure 7.14. (C^1: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.01,\ 0.02,\ 0.03$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

expanding surface. Here we remark that (2.8) clearly holds, and that reducing the relative size of phase 2 is energetically favourable. See Figure 7.18 for the evolution.

A well known phenomenon is the moving of the phase boundary in relation to the neck of a dumbbell for different values of the Gaussian bending rigidities, see e.g. [24, §4.3]. We now demonstrate this behaviour in the sharp interface context. To this end, we choose as initial data a membrane with a neck, and then start an evolution of volume and surface area preserving flow with $\alpha_1^G \in \{-2, 0, 2\}$, while $\alpha_2^G = 0$ and $\varsigma = 9$. For these experiments we choose $\Delta t = 10^{-4}$ and let $\varrho = 1$. See Figure 7.19 for the different evolution. Here we can clearly see that for $\alpha_1^G = 2$ the interface moves down relative to the neck of the dumbbell, while for $\alpha_1^G = -2$ it moves up. Of course, this is due to the neck having negative Gaussian curvature.

In the final experiments we approximate well-known equilibrium shapes from [29, Fig. 8]. To this end, we consider volume and surface area conserving flow for initial surfaces with reduced volumes

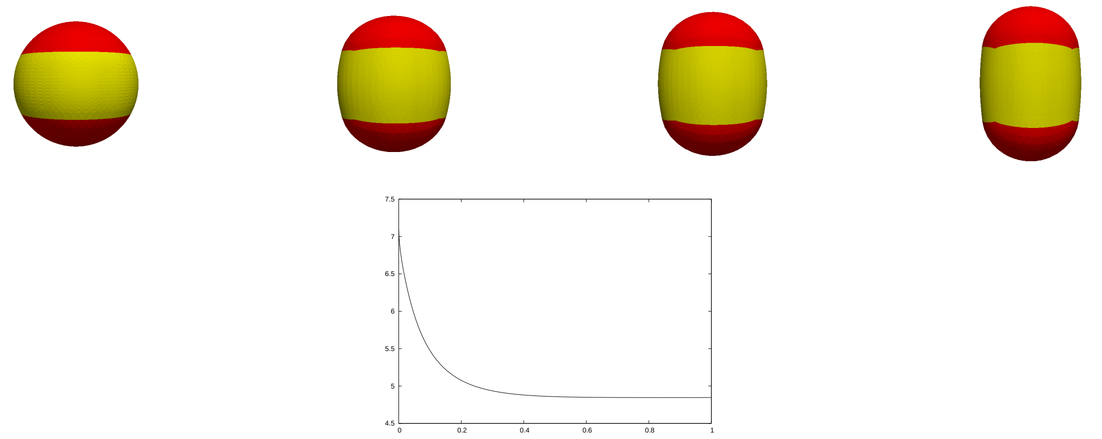

Figure 7.15. (C^1: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$) Surface area preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.2,\ 1$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

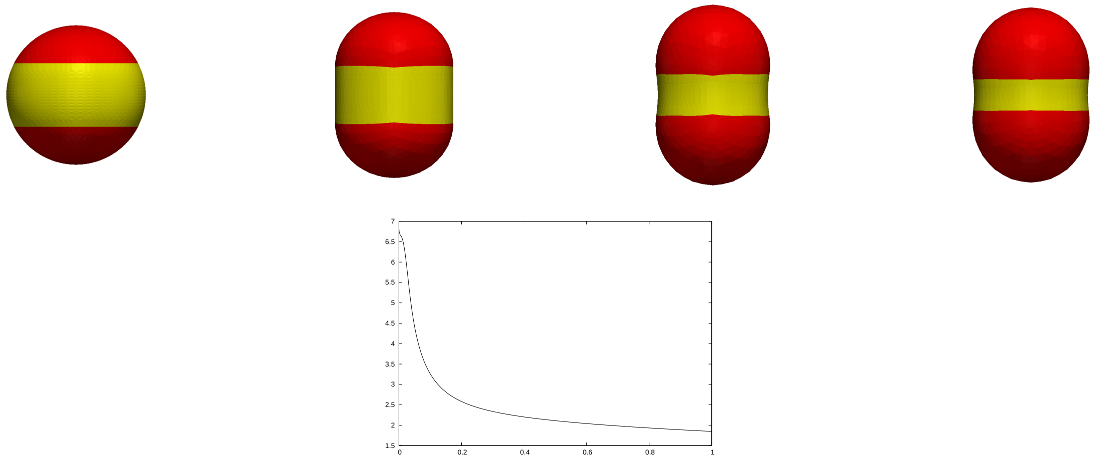

Figure 7.16. (C^1: $\overline{\varkappa}_1 = -2$, $\overline{\varkappa}_2 = -0.5$, $\varsigma = 0.1$, $\varrho = 2$, $\theta = 0.05$) Volume preserving flow. A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.5,\ 1$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

$v_r \in \{0.95,\ 0.91,\ 0.9\}$, where

$$v_r = \frac{3\,\mathcal{L}^3(\Omega^0)}{4\,\pi\,(\frac{\mathcal{H}^2(\Gamma^0)}{4\,\pi})^{\frac{3}{2}}} = \frac{6\,\pi^{\frac{1}{2}}\,\mathcal{L}^3(\Omega^0)}{(\mathcal{H}^2(\Gamma^0))^{\frac{3}{2}}}\,,$$

with Ω^0 denoting the interior of Γ^0. In addition, the two phases are chosen such that they have a surface area ratio of 0.1. See Figure 7.20 for the initial shapes, where in each case we have that the initial discrete surface Γ^0 satisfies $(J_1, J_2) = (2274, 2274)$ and $(K_1, K_2) = (1188, 1188)$ and $\mathcal{H}^2(\Gamma^0) = 4\,\pi$. For these experiments we set $\varsigma = 9$ and $\varrho = 4$. Choosing a time step size of $\Delta t = 10^{-4}$, we integrate the volume and surface area conserving flow to a final time of $t = 0.25$ and report on the obtained shapes in Figure 7.21. These configurations appear to agree well with the computed shapes in [29, Fig. 8].

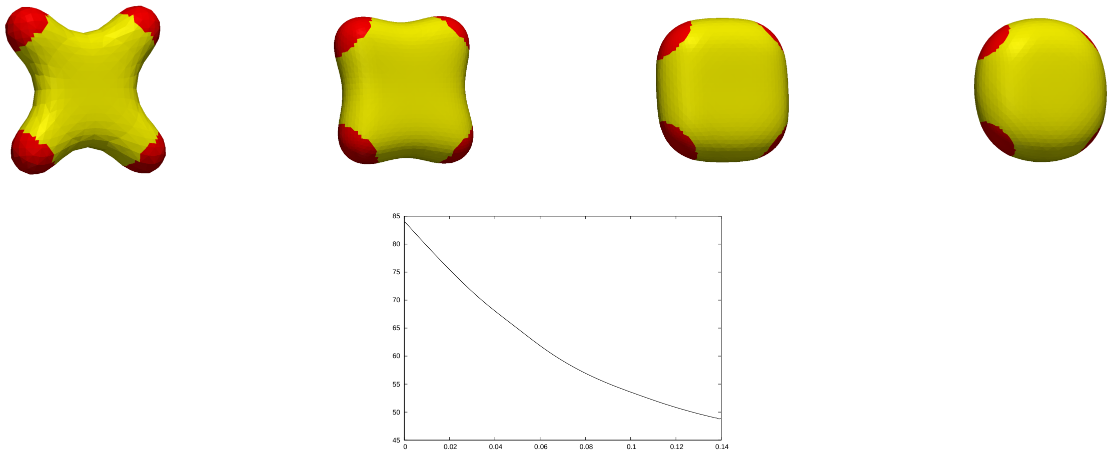

Figure 7.17. (C^1: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.05,\ 0.1,\ 0.14$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

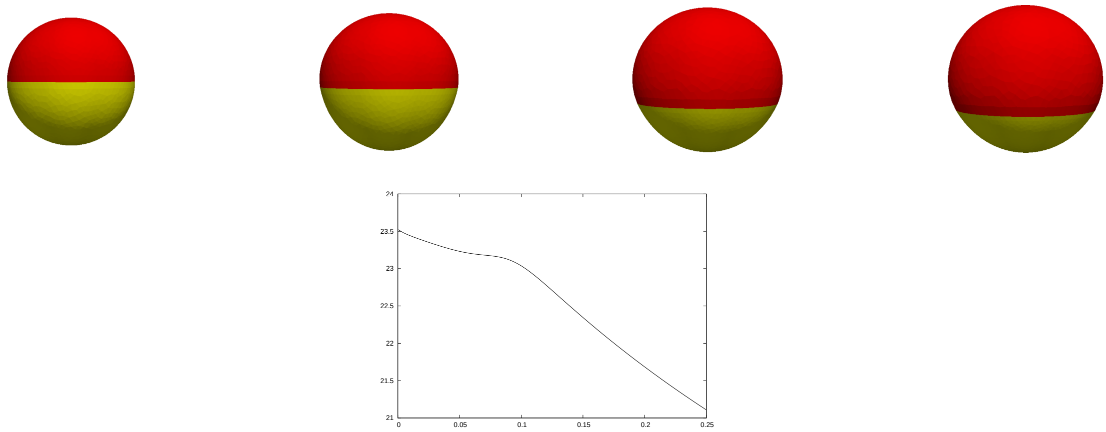

Figure 7.18. (C^1: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = -0.5$, $\varsigma = 0$, $\alpha_1^G = 0.5$, $\alpha_2^G = 1$) A plot of $(\Gamma_i^m)_{i=1}^2$ at times $t = 0,\ 0.1,\ 0.2,\ 0.25$. Below a plot of the discrete energy $E^{m+1}((\Gamma^m)_{i=1}^2)$.

Figure 7.19. (C^1: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 9$, $\alpha_2^G = 0$, $\varrho = 4$) The initial shape on the left, and $(\Gamma_i^m)_{i=1}^2$ at time $t = 0.01$ for $\alpha_1^G = -2$, 0 and 2, respectively.

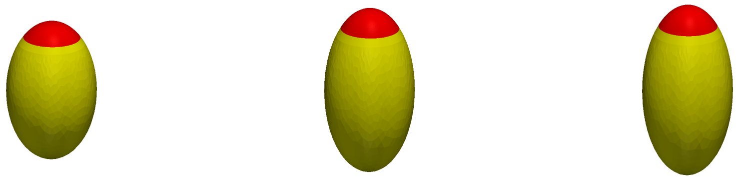

FIGURE 7.20. The initial shapes for $v_r = 0.95$, 0.91 and 0.9, respectively.

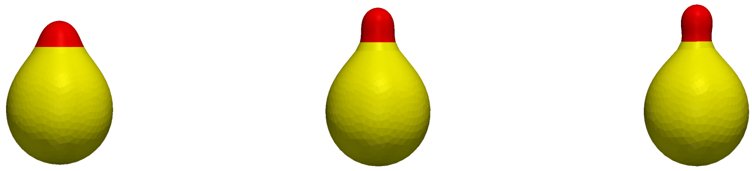

FIGURE 7.21. (C^1: $\overline{\varkappa}_1 = \overline{\varkappa}_2 = 0$, $\varsigma = 9$, $\varrho = 4$) A plot of $(\Gamma_i^m)_{i=1}^2$ at time $t = 0.25$ for the reduced volumes $v_r = 0.95$, 0.91 and 0.9, respectively.

Appendix A. Derivation of strong formulation and boundary conditions

We recall from Section 3 that our numerical method is based on the weak formulation (3.29) and (3.28a–f) of the generalized L^2–gradient flow of the energy $E((\Gamma_i(t))_{i=1}^2)$, see (2.13). It follows from (2.3), (3.28b) and (3.16) that

$$\vec{\varkappa}_i \,.\, \partial_\varepsilon^0 \,\vec{\nu}_i = 0\,, \qquad \partial_\varepsilon^0 \,(\underline{\underline{Q}}_{i,\theta}\,\vec{\varkappa}_i) = -(1-\theta)\,\varkappa_i\,[\nabla_s\,\vec{\chi}]^T\,\vec{\nu}_i\,,$$
$$\tfrac12\left[\alpha_i\,|\vec{\varkappa}_i - \overline{\varkappa}_i\,\vec{\nu}_i|^2 - 2\,\underline{\underline{Q}}_{i,\theta}\,\vec{y}_i\,.\,\vec{\varkappa}_i\right] = -\tfrac12\,\alpha_i\,(\varkappa_i^2 - \overline{\varkappa}_i^2) \quad \text{on} \quad \Gamma_i(t),\ i = 1,2\,.$$

We recall that on the continuous level $\vec{\mathrm{m}}_i = \vec{\mu}_i$ and that $\theta \in [0,1]$ is a fixed parameter. Here we need to choose $\theta = 0$, as otherwise the two conditions in (3.22b,c) are incompatible in general. Then this weak formulation can be formulated as follows. Given $\Gamma_i(0)$, for all $t \in (0,T]$ find $\Gamma_i(t)$ and $\vec{y}_i(t) \in [H^1(\Gamma_i(t))]^d$ such that

$$\begin{aligned}
\left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\Gamma(t)} + \varrho \left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\gamma(t)} &= \sum_{i=1}^2 \Big[\langle \nabla_s\,\vec{y}_i, \nabla_s\,\vec{\chi} \rangle_{\Gamma_i(t)} + \langle \nabla_s\,.\,\vec{y}_i, \nabla_s\,.\,\vec{\chi} \rangle_{\Gamma_i(t)} - \left\langle (\nabla_s\,\vec{y}_i)^T, \underline{\underline{D}}(\vec{\chi})\,(\nabla_s\,\vec{\mathrm{id}})^T \right\rangle_{\Gamma_i(t)} \\
&+ \tfrac12\,\alpha_i \left\langle (\varkappa_i^2 - \overline{\varkappa}_i^2)\,\nabla_s\,\vec{\mathrm{id}}, \nabla_s\,\vec{\chi} \right\rangle_{\Gamma_i(t)} - (1-\theta) \left\langle \varkappa_i\,\vec{y}_i, [\nabla_s\,\vec{\chi}]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)} \Big] - \varsigma \left\langle \vec{\mathrm{id}}_s, \vec{\chi}_s \right\rangle_{\gamma(t)} \\
&+ \sum_{i=1}^2 \alpha_i^G \left[\left\langle \vec{\varkappa}_\gamma\,.\,\vec{\mu}_i, \vec{\mathrm{id}}_s\,.\,\vec{\chi}_s \right\rangle_{\gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_\gamma\,(\vec{\mu}_i)_s, \vec{\chi}_s \right\rangle_{\gamma(t)} \right] \quad \forall\ \vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d\,,
\end{aligned} \tag{A.1}$$

with $\gamma(t) = \partial\Gamma_1(t) = \partial\Gamma_2(t)$,

$$\vec{y}_i = y_i\,\vec{\nu}_i + \vec{u}_i\,, \quad \text{where} \quad y_i = \alpha_i\,(\varkappa_i - \overline{\varkappa}_i) \quad \text{and} \quad \vec{u}_i\,.\,\vec{\nu}_i = 0\,, \quad \text{on } \Gamma_i(t)\,,\ i = 1,2\,. \tag{A.2}$$

Of course, (3.28b) implies that $\vec{u}_i = \vec{0}$ if $\theta \in (0,1]$. Hence, as $\vec{\nu}_i\,.\,[\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i = ([\nabla_s\,\vec{\chi}_i]\,\vec{\nu}_i)\,.\,\vec{\nu}_i = \vec{0}\,.\,\vec{\nu}_i = 0$, it holds that

$$\begin{aligned}
(1-\theta) \left\langle \varkappa_i\,\vec{y}_i, [\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)} &= (1-\theta) \left\langle \varkappa_i\,y_i\,\vec{\nu}_i, [\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)} + (1-\theta) \left\langle \varkappa_i\,\vec{u}_i, [\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)} \\
&= (1-\theta) \left\langle \varkappa_i\,\vec{u}_i, [\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)} = \left\langle \varkappa_i\,\vec{u}_i, [\nabla_s\,\vec{\chi}_i]^T\,\vec{\nu}_i \right\rangle_{\Gamma_i(t)}
\end{aligned} \tag{A.3}$$

for all $\theta \in [0,1]$.

In (A.1) the mean curvatures $\varkappa_i$ are defined by (2.3), the curve curvature vector $\vec{\varkappa}_\gamma$ is given by (2.9), and the conormals $\vec{\mu}_i(t)$ are defined by (2.10) and satisfy $C_1(\vec{\mu}_1 + \vec{\mu}_2) = \vec{0}$. In addition, we have from (3.28c) that

$$\vec{y}_i = -\alpha_i^G \, \vec{\varkappa}_\gamma - C_1 \, \vec{\phi} \quad \text{on} \quad \gamma(t)\,, \quad i = 1,2\,. \tag{A.4}$$

Starting from the weak formulation (A.1), we will now recover the corresponding strong formulation together with the boundary conditions that are enforced by it. It follows from (A.1) and (A.3) that

$$\begin{aligned}
\left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\Gamma(t)} + \varrho \left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\gamma(t)} &= \sum_{i=1}^{2} \Big[\langle \nabla_s (y_i \, \vec{\nu}_i), \nabla_s \, \vec{\chi} \rangle_{\Gamma_i(t)} + \langle \nabla_s \,.\, (y_i \, \vec{\nu}_i), \nabla_s \,.\, \vec{\chi} \rangle_{\Gamma_i(t)} \\
& - \left\langle [\nabla_s (y_i \, \vec{\nu}_i)]^T, \underline{\underline{D}}(\vec{\chi}) \, (\nabla_s \, \vec{\mathrm{id}})^T \right\rangle_{\Gamma_i(t)} + \tfrac{1}{2} \, \alpha_i \left\langle (\varkappa_i^2 - \overline{\varkappa}_i^2), \nabla_s \,.\, \vec{\chi} \right\rangle_{\Gamma_i(t)} + \langle \nabla_s \, \vec{u}_i, \nabla_s \, \vec{\chi} \rangle_{\Gamma_i(t)} \\
& + \langle \nabla_s \,.\, \vec{u}_i, \nabla_s \,.\, \vec{\chi} \rangle_{\Gamma_i(t)} - \left\langle (\nabla_s \, \vec{u}_i)^T, \underline{\underline{D}}(\vec{\chi}) \, (\nabla_s \, \vec{\mathrm{id}})^T \right\rangle_{\Gamma_i(t)} - \left\langle \varkappa_i \, \vec{u}_i, [\nabla_s \, \vec{\chi}]^T \, \vec{\nu}_i \right\rangle_{\Gamma_i(t)} \Big] \\
& + \varsigma \, \langle \vec{\varkappa}_\gamma, \vec{\chi} \rangle_{\gamma(t)} + \sum_{i=1}^{2} \alpha_i^G \left[\left\langle \vec{\varkappa}_\gamma \,.\, \vec{\mu}_i, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_\gamma \, [\vec{\mu}_i]_s, \vec{\chi}_s \right\rangle_{\gamma(t)} \right] \\
& =: \sum_{i=1}^{2} \sum_{\ell=1}^{8} T_\ell^{(i)} + \varsigma \, \langle \vec{\varkappa}_\gamma, \vec{\chi} \rangle_{\gamma(t)} + \sum_{i=1}^{2} \alpha_i^G \left[\left\langle \vec{\varkappa}_\gamma \,.\, \vec{\mu}_i, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\gamma(t)} + \left\langle \underline{\underline{\mathcal{P}}}_\gamma \, [\vec{\mu}_i]_s, \vec{\chi}_s \right\rangle_{\gamma(t)} \right] \\
& \qquad \forall \, \vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d\,.
\end{aligned} \tag{A.5}$$

In order to identify the first term on the right hand side in (A.5), we now recall (A.21) and (A.30) in [10], where we note that in our situation $\beta = 0$, and that the results there are for $d = 3$, but are also true for $d = 2$, where we always assume that $\varsigma = \alpha_1^G = \alpha_2^G = 0$. Hence we have that

$$\begin{aligned}
\sum_{\ell=1}^{8} T_\ell^{(i)} &= \left\langle -\alpha_i \, \Delta_s \, \varkappa_i + \tfrac{1}{2} \, \alpha_i \, (\varkappa_i - \overline{\varkappa}_i)^2 \, \varkappa_i - \alpha_i \, (\varkappa_i - \overline{\varkappa}_i) \, |\nabla_s \, \vec{\nu}_i|^2, \vec{\chi} \,.\, \vec{\nu}_i \right\rangle_{\Gamma_i(t)} + \alpha_i \, \langle (\nabla_s \, \varkappa_i) \,.\, \vec{\mu}_i, \vec{\chi} \,.\, \vec{\nu}_i \rangle_{\gamma(t)} \\
& - \alpha_i \, \langle (\varkappa_i - \overline{\varkappa}_i) \, (\nabla_s \, \vec{\nu}_i) \, \vec{\mu}_i, \vec{\chi} \rangle_{\gamma(t)} - \tfrac{1}{2} \, \alpha_i \left\langle (\varkappa_i - \overline{\varkappa}_i)^2, \vec{\chi} \,.\, \vec{\mu}_i \right\rangle_{\gamma(t)} + B^{(i)} \qquad \forall \, \vec{\chi} \in [H^1_\gamma(\Gamma(t))]^d\,,
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
B^{(i)} &= \langle \nabla_s \,.\, \vec{u}_i, \vec{\chi} \,.\, \vec{\mu}_i \rangle_{\gamma(t)} - \langle \nabla_s \, \vec{u}_i, \vec{\mu}_i \otimes \vec{\chi} \rangle_{\gamma(t)} - \langle \varkappa_i \, \vec{u}_i \,.\, \vec{\mu}_i, \vec{\chi} \,.\, \vec{\nu}_i \rangle_{\gamma(t)} + \langle (\nabla_s \, \vec{u}_i) \, \vec{\mu}_i, (\vec{\chi} \,.\, \vec{\nu}_i) \, \vec{\nu}_i \rangle_{\gamma(t)} \\
&=: \sum_{\ell=1}^{4} D_\ell^{(i)}\,.
\end{aligned} \tag{A.7}$$

It immediately follows from (A.6) that the strong formulation of the flow equation is

$$\vec{\mathcal{V}} = \left[-\alpha_i \, \Delta_s \, \varkappa_i + \tfrac{1}{2} \, \alpha_i \, (\varkappa_i - \overline{\varkappa}_i)^2 \, \varkappa_i - \alpha_i \, (\varkappa_i - \overline{\varkappa}_i) \, |\nabla_s \, \vec{\nu}_i|^2 \right] \vec{\nu}_i \quad \text{on} \quad \Gamma_i(t)\,. \tag{A.8}$$

Collecting the boundary terms arising in (A.5) and (A.6), similarly to [10, (A.32)], gives:

$$\begin{aligned}
\varrho \left\langle \vec{\mathcal{V}}, \vec{\chi} \right\rangle_{\gamma(t)} &= \varsigma \, \langle \vec{\varkappa}_\gamma, \vec{\chi} \rangle_{\gamma(t)} + \sum_{i=1}^{2} \sum_{\ell=1}^{6} B_\ell^{(i)} \\
&= \varsigma \, \langle \vec{\varkappa}_\gamma, \vec{\chi} \rangle_{\gamma(t)} + \sum_{i=1}^{2} \Big[\langle \alpha_i \, (\nabla_s \, \varkappa_i) \,.\, \vec{\mu}_i, \vec{\chi} \,.\, \vec{\nu} \rangle_{\gamma(t)} - \left\langle \tfrac{1}{2} \, \alpha_i \, (\varkappa_i - \overline{\varkappa}_i)^2, \vec{\chi} \,.\, \vec{\mu}_i \right\rangle_{\gamma(t)} \\
& \qquad - \langle \alpha_i \, (\varkappa_i - \overline{\varkappa}_i) \, (\nabla_s \, \vec{\nu}_i) \, \vec{\mu}_i, \vec{\chi} \rangle_{\gamma(t)} + \alpha_i^G \left\langle \vec{\varkappa}_\gamma \,.\, \vec{\mu}_i, \vec{\mathrm{id}}_s \,.\, \vec{\chi}_s \right\rangle_{\gamma(t)}
\end{aligned}$$

$$+\alpha_i^G \left\langle \underline{\underline{\mathcal{P}}}_\gamma\, \vec\chi_s, [\vec\mu_i]_s \right\rangle_{\gamma(t)} + B^{(i)}\Big] \qquad \forall\ \vec\chi \in [H^1_\gamma(\Gamma(t))]^d\,. \tag{A.9}$$

We now investigate the boundary conditions arising from (A.9) in the case $C_1 = 0$. To this end, we recall from (A.2), (A.4) and (2.11) that

$$\alpha_i\,(\varkappa_i - \overline{\varkappa}_i) + \alpha_i^G\, \vec\varkappa_\gamma \,.\, \vec\nu_i = 0 \quad \text{and} \quad \vec u_i = -\alpha_i^G\,(\vec\varkappa_\gamma \,.\, \vec\mu_i)\,\vec\mu_i \quad \text{on} \quad \gamma(t)\,,\ i = 1,2\,. \tag{A.10}$$

Using the simplifications in (A.34)–(A.41) in [10], as well as (A.10), the right hand side of (A.9) can be simplified to obtain

$$\begin{aligned}\varrho \left\langle \vec{\mathcal{V}}, \vec\chi \right\rangle_{\gamma(t)} &= \varsigma\, \langle \vec\varkappa_\gamma, \vec\chi\rangle_{\gamma(t)} + \sum_{i=1}^2 \sum_{\ell=1}^6 B_\ell^{(i)} \\ &= \varsigma\, \langle \vec\varkappa_\gamma, \vec\chi\rangle_{\gamma(t)} + \sum_{i=1}^2 \left\langle (-\tfrac12\,\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2 - \alpha_i^G\,\mathcal{K}_i)\,\vec\mu_i + ((\alpha_i\,(\nabla_s\,\varkappa_i)\,.\,\vec\mu_i - \alpha_i^G\,(\tau_i)_s)\,\vec\nu_i, \vec\chi \right\rangle_{\gamma(t)} \\ &\qquad\qquad \forall\ \vec\chi \in [H^1_\gamma(\Gamma(t))]^d\,. \end{aligned} \tag{A.11}$$

It follows from (A.10) and (A.11) that the necessary boundary conditions are

$$\alpha_i\,(\varkappa_i - \overline{\varkappa}_i) + \alpha_i^G\, \vec\varkappa_\gamma \,.\, \vec\nu_i = 0 \quad \text{on} \quad \gamma(t)\,, \quad i = 1,2\,, \tag{A.12a}$$

$$\varsigma\,\vec\varkappa_\gamma + \sum_{i=1}^2 (-\tfrac12\,\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2 - \alpha_i^G\,\mathcal{K}_i)\,\vec\mu_i + ((\alpha_i\,(\nabla_s\,\varkappa_i)\,.\,\vec\mu_i - \alpha_i^G\,(\tau_i)_s)\,\vec\nu_i = \varrho\,\vec{\mathcal{V}} \quad \text{on} \quad \gamma(t)\,. \tag{A.12b}$$

In the case of surface area preservation, there is an extra term

$$-\sum_{i=1}^2 \lambda_i^A \left\langle \nabla_s\,\vec{\rm id}, \nabla_s\,\vec\chi \right\rangle_{\Gamma_i(t)} = -\sum_{i=1}^2 \lambda_i^A\,\langle \nabla_s\,.\,\vec\chi, 1\rangle_{\Gamma_i(t)} = \sum_{i=1}^2 \lambda_i^A \left[\langle \varkappa_i\,\vec\nu_i, \vec\chi\rangle_{\Gamma_i(t)} - \langle 1, \vec\chi\,.\,\vec\mu_i\rangle_{\gamma(t)} \right]$$

on the right hand side of (A.1), on recalling (2.14), (3.6), (3.11) and (3.25). Similarly, in the case of volume conservation, there is an extra term $-\lambda^V\,\sum_{i=1}^2 \langle \vec\nu_i, \vec\chi\rangle_{\Gamma_i(t)}$ on the right hand side of (A.1), on recalling (2.14), a variational variant of (3.4) and (3.25). Hence overall we obtain

$$\vec{\mathcal{V}}\,.\,\vec\nu_i = -\alpha_i\,\Delta_s\,\varkappa_i + \tfrac12\,\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2\,\varkappa_i - \alpha_i\,(\varkappa_i - \overline{\varkappa}_i)\,|\nabla_s\,\vec\nu_i|^2 + \lambda_i^A\,\varkappa_i - \lambda^V \quad \text{on}\ \Gamma_i(t)\,, \tag{A.13}$$

in place of (A.8), as well as

$$\sum_{i=1}^2 \left[((\alpha_i\,(\nabla_s\,\varkappa_i)\,.\,\vec\mu_i - \alpha_i^G\,(\tau_i)_s)\,\vec\nu_i - (\tfrac12\,\alpha_i\,(\varkappa_i - \overline{\varkappa}_i)^2 + \alpha_i^G\,\mathcal{K}_i + \lambda_i^A)\,\vec\mu_i \right] + \varsigma\,\vec\varkappa_\gamma = \varrho\,\vec{\mathcal{V}} \quad \text{on} \quad \gamma(t)\,, \tag{A.14}$$

in place of (A.12b).

We now investigate the boundary conditions arising from (A.9) in the case $C_1 = 1$, where we recall that in this situation $\vec\nu = \vec\nu_1 = \vec\nu_2$ and $\vec\mu = \vec\mu_2 = -\vec\mu_1$ on $\gamma(t)$. To this end, we obtain from (A.2), (A.4) and (2.11) that $\vec y_2 - \vec y_1 = -(\alpha_2^G - \alpha_1^G)\,\vec\varkappa_\gamma$ on $\gamma(t)$, and hence

$$\alpha_1\,(\varkappa_1 - \overline{\varkappa}_1) + \alpha_1^G\,\vec\varkappa_\gamma\,.\,\vec\nu = \alpha_2\,(\varkappa_2 - \overline{\varkappa}_2) + \alpha_2^G\,\vec\varkappa_\gamma\,.\,\vec\nu \quad \text{on} \quad \gamma(t)\,, \tag{A.15a}$$

$$\vec u_2 - \vec u_1 = -(\alpha_2^G - \alpha_1^G)\,(\vec\varkappa_\gamma\,.\,\vec\mu)\,\vec\mu \quad \text{on} \quad \gamma(t)\,. \tag{A.15b}$$

We now rewrite some of the terms in (A.9). To this end, we first note that it follows from (2.11), (2.4) and (2.17) that

$$\vec\varkappa_\gamma\,.\,\vec\nu_i = \vec{\rm id}_{ss}\,.\,\vec\nu_i = -\vec{\rm id}_s\,.\,[\vec\nu_i]_s = \mathrm{II}_i(\vec{\rm id}_s, \vec{\rm id}_s) \quad \text{and} \quad \varkappa_i = \mathrm{II}_i(\vec{\rm id}_s, \vec{\rm id}_s) + \mathrm{II}_i(\vec\mu_i, \vec\mu_i) \quad \text{on} \quad \gamma(t)\,. \tag{A.16}$$

Moreover, it follows from (2.18) and (2.10) that

$$[\vec\nu_i]_s \times \vec{\rm id}_s = -\tau_i\,\vec\mu_i \times \vec{\rm id}_s = (-1)^i\,\tau_i\,\vec\nu_i \qquad \text{on}\ \gamma(t)\,, \tag{A.17}$$

where we have observed that $[\vec\nu_i]_s$ is perpendicular to $\vec\nu_i$. We also note from (A.17) and (2.10) that

$$[\vec\mu_i]_s = (-1)^i \left([\vec\nu_i]_s \times \vec{\rm id}_s + \vec\nu_i \times \vec{\rm id}_{ss}\right) = \tau_i\,\vec\nu_i + (-1)^i\,(\vec\varkappa_\gamma\,.\,\vec\mu_i)\,\vec\nu_i \times \vec\mu_i = \tau_i\,\vec\nu_i - (\vec\varkappa_\gamma\,.\,\vec\mu_i)\,\vec{\rm id}_s \quad \text{on } \gamma(t)\,. \tag{A.18}$$

Now it follows from (2.17) that

$$(\nabla_s\,\vec\nu_i)\,\vec\mu_i\,.\,\vec\chi = -\mathrm{I\!I}_i(\vec\mu_i,\vec{\rm id}_s)\,\vec\chi\,.\,\vec{\rm id}_s - \mathrm{I\!I}_i(\vec\mu_i,\vec\mu_i)\,\vec\chi\,.\,\vec\mu_i\,,$$

and so

$$\begin{aligned} B_3^{(i)} &= \alpha_i \left\langle (\varkappa_i - \overline\varkappa_i), \mathrm{I\!I}_i(\vec\mu_i,\vec{\rm id}_s)\,\vec\chi\,.\,\vec{\rm id}_s + \mathrm{I\!I}_i(\vec\mu_i,\vec\mu_i)\,\vec\chi\,.\,\vec\mu_i \right\rangle_{\gamma(t)} \\ &= \alpha_i \left\langle (\varkappa_i - \overline\varkappa_i), \tau_i\,\vec\chi\,.\,\vec{\rm id}_s + (\varkappa_i - \vec\varkappa_\gamma\,.\,\vec\nu_i)\,\vec\chi\,.\,\vec\mu_i \right\rangle_{\gamma(t)}\,, \end{aligned} \tag{A.19}$$

where we have noted (2.18) and (A.16). It follows from (A.36) and (A.37) in [10] that

$$\begin{aligned} B_4^{(i)} + B_5^{(i)} &= \alpha_i^G \left\langle [\vec\varkappa_\gamma\,.\,\vec\nu_i\,\tau_i - (\vec\varkappa_\gamma\,.\,\vec\mu_i)_s]\,\vec{\rm id}_s + \left[\tau_i^2 - (\vec\varkappa_\gamma\,.\,\vec\mu_i)^2\right]\vec\mu_i, \vec\chi \right\rangle_{\gamma(t)} \\ &\quad - \alpha_i^G \left\langle [(\tau_i)_s + (\vec\varkappa_\gamma\,.\,\vec\mu_i)\,\vec\varkappa_\gamma\,.\,\vec\nu_i]\,\vec\nu_i, \vec\chi \right\rangle_{\gamma(t)}\,. \end{aligned} \tag{A.20}$$

We have from (A.7) above and (A.39) in [10] that

$$D_1^{(i)} + D_2^{(i)} = \left\langle (\vec u_i)_s, (\vec\chi\,.\,\vec\mu_i)\,\vec{\rm id}_s - (\vec\chi\,.\,\vec{\rm id}_s)\,\vec\mu_i \right\rangle_{\gamma(t)}\,. \tag{A.21}$$

As $\vec u_i\,.\,\vec\nu_i = 0$, we have from (A.7), (2.17) and (A.16) that

$$\begin{aligned} D_3^{(i)} + D_4^{(i)} &= -\left\langle \varkappa_i\,\vec\mu_i + (\nabla_s\,\vec\nu_i)\,\vec\mu_i, (\vec\chi\,.\,\vec\nu_i)\,\vec u_i \right\rangle_{\gamma(t)} \\ &= \left\langle \mathrm{I\!I}_i(\vec\mu_i,\vec{\rm id}_s), (\vec u_i\,.\,\vec{\rm id}_s)\,\vec\chi\,.\,\vec\nu_i \right\rangle_{\gamma(t)} + \left\langle \mathrm{I\!I}_i(\vec\mu_i,\vec\mu_i) - \varkappa_i, (\vec u_i\,.\,\vec\mu_i)\,\vec\chi\,.\,\vec\nu_i \right\rangle_{\gamma(t)} \\ &= \left\langle \tau_i\,\vec u_i\,.\,\vec{\rm id}_s - (\vec\varkappa_\gamma\,.\,\vec\nu_i)\,\vec u_i\,.\,\vec\mu_i, \vec\chi\,.\,\vec\nu_i \right\rangle_{\gamma(t)}\,. \end{aligned} \tag{A.22}$$

Therefore (A.7), (A.21) and (A.22) yield that

$$\begin{aligned} \sum_{i=1}^2 B_6^{(i)} &= \left\langle (\vec u_2 - \vec u_1)_s, (\vec\chi\,.\,\vec\mu)\,\vec{\rm id}_s - (\vec\chi\,.\,\vec{\rm id}_s)\,\vec\mu \right\rangle_{\gamma(t)} + \left\langle (\vec u_2 - \vec u_1)\,.\,\vec{\rm id}_s, \tau\,(\vec\chi\,.\,\vec\nu) \right\rangle_{\gamma(t)} \\ &\quad - \left\langle (\vec u_2 - \vec u_1)\,.\,\vec\mu (\vec\varkappa_\gamma\,.\,\vec\nu)\,\vec\chi\,.\,\vec\nu \right\rangle_{\gamma(t)}\,, \end{aligned}$$

where $\tau = \tau_2 = -\tau_1$. Hence we obtain from (A.15b) and (A.18) that

$$\sum_{i=1}^2 B_6^{(i)} = [\alpha_i^G]_1^2 \left\langle (\vec\varkappa_\gamma\,.\,\vec\mu)_s\,\vec{\rm id}_s + (\vec\varkappa_\gamma\,.\,\vec\mu)^2\,\vec\mu + (\vec\varkappa_\gamma\,.\,\vec\mu)\,(\vec\varkappa_\gamma\,.\,\vec\nu)\,\vec\nu, \vec\chi \right\rangle_{\gamma(t)}\,. \tag{A.23}$$

Combining (A.9), (A.19), (A.20) and (A.23) yields, on noting (A.15a) and (2.11), that

$$\begin{aligned} \varrho \left\langle \vec{\mathcal V}, \vec\chi \right\rangle_{\gamma(t)} &= \varsigma \left\langle \vec\varkappa_\gamma, \vec\chi \right\rangle_{\gamma(t)} + \sum_{i=1}^2 \sum_{\ell=1}^6 B_\ell^{(i)} = \left\langle [\alpha_i\,(\nabla_s\,\varkappa_i)]_1^2\,.\,\vec\mu - [\alpha_i^G]_1^2\,\tau_s + \varsigma\,\vec\varkappa_\gamma\,.\,\vec\nu, \vec\chi\,.\,\vec\nu \right\rangle_{\gamma(t)} \\ &\quad + \left\langle -\tfrac12\,[\alpha_i\,(\varkappa_i - \overline\varkappa_i)^2]_1^2 + [\alpha_i\,(\varkappa_i - \overline\varkappa_i)\,(\varkappa_i - \vec\varkappa_\gamma\,.\,\vec\nu)]_1^2 + [\alpha_i^G]_1^2\,\tau^2 + \varsigma\,\vec\varkappa_\gamma\,.\,\vec\mu, \vec\chi\,.\,\vec\mu \right\rangle_{\gamma(t)} \\ &\qquad \forall\ \vec\chi \in [H^1_\gamma(\Gamma(t))]^d\,. \end{aligned}$$

This yields the boundary conditions

$$[\alpha_i\,(\nabla_s\,\varkappa_i)]_1^2\,.\,\vec\mu - [\alpha_i^G]_1^2\,\tau_s + \varsigma\,\vec\varkappa_\gamma\,.\,\vec\nu = \varrho\,\vec{\mathcal V}\,.\,\vec\nu \quad \text{on } \gamma(t)\,, \tag{A.24a}$$

$$-\tfrac12\,[\alpha_i\,(\varkappa_i - \overline\varkappa_i)^2]_1^2 + [\alpha_i\,(\varkappa_i - \overline\varkappa_i)\,(\varkappa_i - \vec\varkappa_\gamma\,.\,\vec\nu)]_1^2 + [\alpha_i^G]_1^2\,\tau^2 + \varsigma\,\vec\varkappa_\gamma\,.\,\vec\mu = \varrho\,\vec{\mathcal V}\,.\,\vec\mu \quad \text{on } \gamma(t)\,, \tag{A.24b}$$

as well as $\varrho\,\vec{\mathcal{V}}\,.\,\vec{\rm id}_s = 0$ on $\gamma(t)$, which has no effect on the evolution of $(\Gamma_i(t))_{i=1}^2$. Clearly, (A.15a) and (A.24a,b) yield the conditions (2.20a–c), on accounting for the surface area constraints analogously to the case $C_1 = 0$.

Acknowledgements

The authors gratefully acknowledge the support of the Regensburger Universitätsstiftung Hans Vielberth.

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