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Methods of Reflections: relations with Schwarz methods and classical stationary iterations, scalability and preconditioning.

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Abstract. The basic idea of the method of reflections appeared almost two hundred years ago; it is a method of successive approximations for the interaction of particles within a fluid, and it seems intuitively related to the Schwarz domain decomposition methods, the subdomains being the complements of the particle domains. We show in this paper that indeed there is a direct correspondence between the methods of reflections and Schwarz methods in the two particle/subdomain case. This allows us to give a new convergence analysis based on maximum principle techniques with precise convergence estimates that one could not obtain otherwise. We then show however also that in the case of more than two particles/subdomains, the methods of reflections and the Schwarz methods are really different methods, with different convergence properties. Using substructuring techniques from domain decomposition, we then show that the methods of reflections are classical block Jacobi and block Gauss-Seidel methods for the interface traces, and we derive new, relaxed versions of the methods of reflections with better convergence properties. We finally also introduce for the first time coarse corrections for the methods of reflections to make them scalable in the case when the number of particles becomes large. The substructured formulations permit the easy use of the methods of reflections as preconditioners for Krylov methods, and we illustrate scalability and preconditioning properties with numerical experiments.

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1. Introduction

We start by briefly tracing the invention of the method of reflections and its historical development, for a more detailed treatment, see [11]. The fundamental idea for the method of reflections can already be found in the book of Murphy [45, p. 93] from 1833 under the name “principle of successive influences”, which clearly indicates that the method of reflections is a method of successive approximations. Lamb used in 1906 a similar approach for Laplace’s equation [37, p. 122]. A related method in the work of Lorentz [41, p. 29] was later called “Spiegelungsmethode” (method of reflections) in [28, p. 928]. The method of reflections itself was then presented in concrete mathematical notation by Smoluchowski in 1911 with the goal to understand how the motion of a sphere in a viscous fluid is influenced by the presence or motion of another sphere [49]. In the case of the Stokes equation, Smoluchowski assumes that the radii of the two spheres are small compared to their distance, and then uses the method of reflections to compute a series expansion of the coupled solution up to some order in the inverse distance of the spheres. The method is thus a direct method to obtain a series solution in the inverse

of the distance, up to some order. Smoluchowski then generalizes the method of reflections to the case of more than two spheres, leading to similar series approximations. In 1934, Golusin introduced a parallel method of reflections for Laplace's equation for J objects [25, 24], and derived a condition for its convergence, which indicates that the parallel method of reflections in the case of more than two objects converges only under certain additional conditions. Golusin conjectured that these conditions depend on the distance between the objects and their radius. In 1942, Burgers [4, 5] investigated the influence of the concentration of spherical particles on the sedimentation velocity for the Stokes equation, mentioning the work of Smoluchowski, but without describing precisely an algorithm, and using to a large extent physical intuition. In 1959, Kynch presented for the Stokes equation the idea of simply summing two solutions corresponding to two particles alone in [36, p. 197], under the assumption that the distance between their centers is again large. This could be interpreted as a parallel method of reflections, where the separate contributions are also summed, but again, no general algorithm is given. Happel and Brenner explained in 1983 a different parallel version of the method of reflections which alternates between one fixed object and the group of all the others treated in parallel, see [27], with the goal to increase the order of approximation of the expansion of the solution in a neighborhood of a given object. Their method has to be applied (independently) for each particle.

Luke gave then in 1989 a first convergence analysis for the alternating method of reflections applied to the Stokes equation [42], using a variational characterization of the method based on projections, similar to one of the classical convergence analyses of the Schwarz method given by Lions in the first of his three seminal papers [39]. Kim and Karrila dedicated an entire chapter in their book from 1991 to the parallel method of reflections for the Stokes problem [34]. The method is first motivated like in [36] by just summing two one-particle solutions, and only the first terms in the series expansion are obtained. Dhont also dedicated a special section to the alternating method of reflections for the Stokes equation in 1996 [14, Section 5.12]. The case of two objects is first treated, and then an extension to three objects is given, where Dhont goes cyclically through the three object in the algorithm. Balabane and Tirel proved in 1997 convergence of the alternating method of reflections for the Helmholtz equation in unbounded domains in [3]. In 1998, Tirel generalized these results in her thesis [50] and proved conditional convergence of the parallel method of reflections; see also [2]. These convergence results are valid however only in low frequency regimes. Similar results are obtained in [12], where multiple scattering problems are studied; see also [22, 53] for numerical applications, the book [48] for the numerical treatment of boundary integral equations using boundary element methods, and, e.g., [43, 44] for theoretical studies on multiple scattering problems. In 2001, Ichiki and Brady presented the parallel method of reflections [31] for Stokes type problems. They started with the two particle case, and then gave a generalization just summing all contributions that were computed in parallel. They presented this iterative approach also in matrix form, relating it to a stationary iteration based on a matrix splitting. By numerical experiments, they showed that the method does not converge for three particles, if the separation distance of the particles is not large enough. They thus proposed to use the method as a preconditioner for a Krylov method. Traytak posed in 2006 in a short note directly the parallel method of reflections for N objects, written in PDE form for Laplace's equation [52, Section 2], and then used a theorem proved by Golusin [24] to derive sufficient conditions for the convergence based on the distances between the objects. More recently, Höfer and Velázquez used the parallel method of reflections also as an analytic tool to prove homogenization results [30] (see also [32] and [29]), and they modified the usual parallel method by adding different weighting coefficients. Since they were interested in the theoretical case of an infinite number of objects, they could not use an alternating method. Laurent, Legendre and Salomon studied the alternating and parallel methods of reflections in [38] for various types of boundary conditions, introducing also an averaged version of the parallel method. They proved convergence based on the alternating projection method in Hilbert spaces, see for example [46], and also using techniques like in [3, 2].

So there are two main variants of the method of reflections: the alternating one and the parallel one. There are also two different approaches to analyze the convergence of the method of reflections: first, people worked on direct estimates performed on the single/double layer formulation of the boundary value problems involved in the iterations, see [24, 52, 3, 2]. There is however also the interpretation of the method as alternating projections in Hilbert spaces, see [42, 38]. In the case of orthogonal projections this interpretation leads to convergence estimates.

When we started studying methods of reflections more than three years ago, we thought immediately that the methods must be intimately related to the Schwarz domain decomposition methods. This intuition was confirmed when we studied the literature and found that analysis techniques based on projections were already used to study methods of reflections like for Schwarz methods, and such a possible relation was even mentioned in the literature (“This paper considers a reflection method in the spirit of Schwarz’s alternating procedure” [42]). We will show here that the methods of reflections can be indeed identified with Schwarz domain decomposition methods in the case of two particles/subdomains, which leads to a new convergence proof with sharp convergence estimates using maximum principle techniques. In the case of many particles/subdomains however the methods of reflections are fundamentally different from Schwarz domain decomposition methods. The main difference is that in domain decomposition, the interface data is to be determined by the subdomain iteration, whereas in the method of reflections, the interface data is given on the boundary of the particles by the problem. Substructuring techniques from domain decomposition allow us however to reformulate the methods of reflections in the form of standard block Gauss-Seidel and block Jacobi iterations in the traces for integral formulations. This reformulation leads to a new understanding of the method of reflections also for many particles, and reveals shortcomings, especially in the parallel variant. It also allows us to develop new coarse space corrections, again using domain decomposition techniques, in order to obtain scalable methods of reflections when the number of particles grows.

Our paper is structured as follows: we start by presenting in Section 2 the class of Laplace problems on perforated domains we will use to understand the methods of reflections. We then present the different forms of the methods of reflections: in Section 3 the alternating method of reflections, in Section 4 the parallel method of reflections, in Section 5 some other variants of the method of reflections. We give in each case reformulations of the methods using domain decomposition techniques, which allows us to study their convergence properties and give new convergence proofs and convergence estimates. In Section 6, we then give a scalability analysis, and also propose for the first time for the methods of reflections a coarse space correction to make them scalable in the case when the number of objects becomes large. We show numerical experiments in Section 7, followed by a brief conclusion in Section 8.

2. The Laplace problem in perforated domains

We consider a Laplace problem where J objects O_j (holes) are present in a simply connected domain $\Omega \subset \mathbb{R}^n$; for an example, see Figure 2.1 on the left. We assume that Ω and O_j are simply connected with sufficiently smooth boundaries $\partial\Omega$ and ∂O_j , e.g., of class C^1 , and that

$$\overline{O_j} \cap \overline{O_k} = \emptyset \text{ for any } j \neq k. \quad (2.1)$$

Our goal is finding the (weak) solution $u \in H^1(\Omega \setminus \cup_j \overline{O_j})$ to the problem

$$\Delta u = 0 \text{ in } \Omega \setminus \cup_j \overline{O_j}, \quad u = 0 \text{ on } \partial\Omega, \quad u = g_j \text{ on } \partial O_j \text{ for } j = 1, \dots, J, \quad (2.2)$$

where we assume that the functions g_j are bounded on ∂O_j and in $H^{1/2}(\partial O_j)$ for $j = 1, \dots, J$. This problem is well-posed; see, e.g., [23, 35]. In order to introduce the method of reflections, we modify (2.2) by extending harmonically the solution u into the objects O_j . We define $\tilde{\Omega} := \Omega \setminus \cup_{j=1}^J \partial O_j$

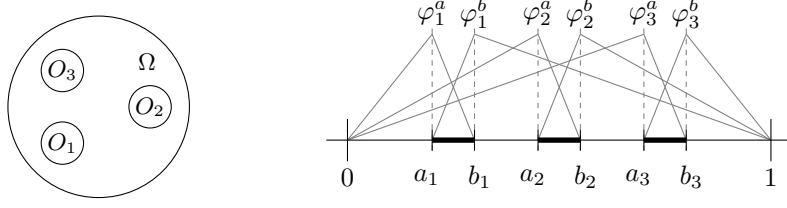


FIGURE 2.1. Left: example of a domain Ω (a disk) with three objects O_j (holes). Right: example of the functions φ_j^a and φ_j^b for $J = 3$.

and then (2.2) becomes

$$\Delta u = 0 \text{ in } \tilde{\Omega}, \quad u = 0 \text{ on } \partial\Omega, \quad u = g_j \text{ on } \partial O_j \text{ for } j = 1, \dots, J. \quad (2.3)$$

The solution u to (2.3) belongs to the Hilbert space

$$H := \{v \in H_0^1(\Omega) : \Delta v = 0 \text{ in } \tilde{\Omega}\}, \quad (2.4)$$

endowed with the H_0^1 scalar product $\langle \cdot, \cdot \rangle_{H_0^1} := \langle \nabla(\cdot), \nabla(\cdot) \rangle_{L^2}$, and by Weyl's theorem [23], which states that every weak solution to the Laplace equation is a smooth solution, we have that $H \subset C^\infty(\tilde{\Omega})$.

We introduce the subspaces H_j of H defined by

$$H_j := \{v \in H : \Delta v = 0 \text{ in } \Omega \setminus \partial O_j\}. \quad (2.5)$$

If Ω is unbounded, one needs to consider the completion of $H_0^1(\Omega)$ for the semi-norm induced by $\langle \cdot, \cdot \rangle_{H_0^1(\Omega)}$, for instance weighted Sobolev spaces, see [13, 26, 1], and similar results could be obtained.

It has been proved in [38] that $H = \sum_{j=1}^J H_j$. For our purposes, we need to refine this result. We have the following lemma (a similar result for a different type of equation can be found in [12]; see also [44, 43] for other theoretical studies).

Lemma 2.1. *Under the Assumption (2.1), we have that $H = \bigoplus_{j=1}^J H_j$, which means that the subspaces H_j are linearly independent, that is $(H_1 + \dots + H_{j-1}) \cap H_j = \{0\}$ for any $j \geq 2$.*

Proof. Consider any function v in $H_j \cap H_k$ for any $j \neq k$. Then, v is harmonic everywhere and zero on $\partial\Omega$, thus $v = 0$. This and the fact that $H = \sum_{j=1}^J H_j$ proved in [38] imply the claim. \blacksquare

To explain Lemma 2.1 we consider a one-dimensional example, and then show that if Assumption (2.1) is not satisfied, then the result does not hold in general. The domain Ω is the interval $(0, 1)$ and the objects are subintervals $O_j = (a_j, b_j)$ for $1 \leq j \leq J$ such that $0 < a_1 < b_1 < a_2 < \dots < a_J < b_J < 1$. In this case, the space H_j is spanned by the hat-functions

$$\varphi_j^a(x) := \begin{cases} \frac{x}{a_j} & \text{if } x \in [0, a_j], \\ \frac{b_j - x}{b_j - a_j} & \text{if } x \in [a_j, b_j], \\ 0 & \text{if } x \in [b_j, 1], \end{cases} \quad \varphi_j^b(x) := \begin{cases} 0 & \text{if } x \in [0, a_j], \\ \frac{x - a_j}{b_j - a_j} & \text{if } x \in [a_j, b_j], \\ \frac{1 - x}{1 - b_j} & \text{if } x \in [b_j, 1], \end{cases}$$

that have value 1 at a_j and b_j and are supported in $[0, b_j]$ and $[a_j, 1]$, for an example, see Figure 2.1 (right). Notice that $\dim H_j = 2$. Therefore, since all the points a_j and b_j are distinct, we have that all the functions φ_j^a and φ_j^b are linearly independent, $H = \text{span}\{\varphi_1^a, \varphi_1^b, \dots, \varphi_J^a, \varphi_J^b\}$, and $\dim H = 2J$. The hypothesis $a_j < b_j < a_{j+1} < b_{j+1}$ then clearly implies that $(H_1 + \dots + H_{k-1}) \cap H_k = \{0\}$ for $2 \leq k \leq J$, which is the result of Lemma 2.1. On the other hand, if we assume that two objects are not distinct, that is, e.g., $b_j = a_{j+1}$, then one can verify that $c_0\varphi_{j+1}^a + c_1\varphi_{j+1}^b + c_2\varphi_j^a + c_3\varphi_j^b = 0$, with $c_0 = 1$, $c_1 = \frac{1-b_{j+1}}{1-b_j}$, $c_2 = -\frac{a_j}{a_{j+1}}$, and $c_3 = -1$. Hence the functions φ_j^a , φ_j^b , φ_{j+1}^a , and φ_{j+1}^b are not

linearly independent and $\dim H = 2J - 1 < 2J = \sum_{j=1}^J \dim H_j$. Hence H cannot be written as the direct sum of the subspaces H_j .

Iterative methods suitable for solving problem (2.2)-(2.3) sequentially or in parallel are the methods of reflections (MR), which we will present and study in the following sections.

3. The alternating method of reflections

We now present the alternating method of reflections (AltMR), give a new volume and substructured formulation and investigate the relation with block Gauss-Seidel and the alternating Schwarz method.

3.1. The classical AltMR formulations

The alternating method of reflections was invented for two objects by Smoluchowski in 1911 [49], and then extended to the general case of J objects by Luke [42]. Luke explains the method as follows: "The strategy of the method of reflections is to repeatedly correct the boundary values on the various objects. The reflection procedure is formalized as follows": it starts with $u^0 \in H$, that is

$$\Delta u^0 = 0 \text{ in } \Omega \setminus \cup_j \partial O_j, \quad u^0 = 0 \text{ on } \partial \Omega. \quad (3.1)$$

Note that u^0 does not satisfy the boundary conditions of (2.2) on the holes, which would require $u^0 = g_j$ on ∂O_j for $j = 1, \dots, J$. The sequence of approximate solutions $\{u^k\}_{k \in \mathbb{N}}$ is defined as follows: given u^0 , one computes for iteration index $k = 1, 2, \dots$ from u^{k-1}

$$u^{k-1+\frac{1}{J}} = u^{k-1} + d_1^k, \quad u^{k-1+\frac{2}{J}} = u^{k-1} + d_1^k + d_2^k, \quad u^{k-1+\frac{3}{J}} = u^{k-1} + d_1^k + d_2^k + d_3^k, \quad \dots$$

and then obtains the new approximation

$$u^k = u^{k-1} + \sum_{j=1}^J d_j^k, \quad (3.2)$$

where $d_j^k \in H_j$ is computed in such a way that, when added to $u^{k-1+\frac{j-1}{J}}$, the value on the boundary ∂O_j is corrected to its given boundary value g_j , that is $u^{k-1+\frac{j}{J}} = u^{k-1+\frac{j-1}{J}} + d_j^k = g_j$ on ∂O_j . This means that d_j^k , for $j = 1, \dots, J$, must be the solution to

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial \Omega, \quad d_j^k = g_j - u^{k-1+\frac{j-1}{J}} = g_j - u^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k \text{ on } \partial O_j. \quad (3.3)$$

A simple and intuitive explanation of the iteration described in (3.2)-(3.3) can be found in [11].

In [38, 3, 11] an equivalent form of (3.3) is presented, which we derive now for completeness: we consider the boundary condition of (3.3) on ∂O_j , for $j = 1, \dots, J$, and manipulate it as follows:

$$d_j^k = g_j - u^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k = g_j - u^{k-2} - \sum_{\ell=1}^j d_\ell^{k-1} - \sum_{\ell=j+1}^J d_\ell^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k.$$

Now, we notice that $g_j - u^{k-2} - \sum_{\ell=1}^j d_\ell^{k-1} = 0$, since d_j^{k-1} solves at iteration $k-1$ problem (3.3). Therefore, we have obtained that d_j^k can be expressed on ∂O_j as combinations of other differences only,

$$d_j^k = - \sum_{\ell=j+1}^J d_\ell^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k, \quad (3.4)$$

and the explicit dependence on g_j and u^{k-1} disappeared. Hence, iteration (3.3) becomes

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial \Omega, \quad d_j^k = - \sum_{\ell=1}^{j-1} d_\ell^k - \sum_{\ell=j+1}^J d_\ell^{k-1} \text{ on } \partial O_j, \quad (3.5)$$

for $j = 1, \dots, J$, that is the form of the sequential method of reflections presented in [38, 3]. Now the sequences $\{d_j^k\}_{k \in \mathbb{N}^+}$ have to be initialized for all j . To this end, it is sufficient, for example, to consider (3.3) for $k = 1$ and $j = 1, \dots, J$:

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^1 = 0 \text{ on } \partial \Omega, \quad d_j^1 = g_j - \sum_{\ell=1}^{j-1} d_\ell^1 - u^0 \text{ on } \partial O_j. \quad (3.6)$$

Notice that the AltMR iteration in the form (3.3) cannot be initialized with an arbitrary function: the initial guess u^0 must be in H (recall (3.1)). A choice of a function $u^0 \notin H$ will produce a sequence $\{u^k\}_k$ that does not necessarily converge to the solution u to (2.2). In fact, (3.3) produces corrections d_j^k which are harmonic in $\Omega \setminus \cup_{j=1}^J \partial O_j$. Hence, if u^0 can be decomposed into the sum of harmonic and non-harmonic components, namely $u^0 = u_{\text{harm}}^0 + u_{\text{non-harm}}^0$, then only u_{harm}^0 is corrected by the MR procedure. The AltMR as a program is given in Algorithm 1.

Algorithm 1 Alternating Method of Reflections (AltMR)

Require: K (maximum number of iterations), tol (tolerance).

- 1: Set $u^0 \in H$ and $k = 1$.
 - 2: **for** $j = 1: J$ **do**
 - 3: Initialize d_j^1 solving problem (3.6).
 - 4: **end for**
 - 5: Compute $u^1 = u^0 + \sum_{j=1}^J d_j^1$.
 - 6: **while** $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
 - 7: Update $k = k + 1$.
 - 8: **for** $j = 1: J$ **do**
 - 9: Compute d_j^k solving problem (3.5).
 - 10: **end for**
 - 11: Compute the approximation $u^k = u^{k-1} + \sum_{j=1}^J d_j^k$.
 - 12: **end while**
-

3.2. A new formulation of the AltMR

To formulate the AltMR in a third equivalent way, we recall the d_j^k that solve (3.3), define J sequences $\{v_j^k\}_{k \in \mathbb{N}}$ for any $j \in \{1, \dots, J\}$ as

$$v_j^k := v_j^{k-1} + d_j^k \text{ with } v_j^0 \in H_j, \quad (3.7)$$

and $u_0 := \sum_{j=1}^J v_j^0$ (recall Lemma 2.1). Notice that since $d_j^k \in H_j$, $v_j^k \in H_j$ as well. Now, we notice that (3.2) implies that $u^k = u_0 + \sum_{j=1}^J \sum_{n=0}^k d_j^n$, which allows us to compute

$$u^k = u_0 + \sum_{j=1}^J \sum_{n=1}^k d_j^n = u_0 + \sum_{j=1}^J \sum_{n=0}^k (v_j^n - v_j^{n-1}) = u_0 + \sum_{j=1}^J (v_j^k - v_j^0),$$

and recalling that $\sum_{j=1}^J v_j^0 = u_0$ we obtain

$$u^k = \sum_{j=1}^J v_j^k. \quad (3.8)$$

Notice that equation (3.8) allows us to express the approximate solution at the k th iteration as a decomposition (unique according to Lemma 2.1) into J components, each of them being associated with an object and belonging to one of the spaces H_j . Starting from (3.7), using (3.8) and the boundary condition in (3.3), that is $d_j^k = g_j - u^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k$ on ∂O_j , we have on ∂O_j that

$$v_j^k = v_j^{k-1} + d_j^k = v_j^{k-1} + g_j - u^{k-1} - \sum_{\ell=1}^{j-1} (v_\ell^k - v_\ell^{k-1}) = g_j - \sum_{\ell=j+1}^J v_\ell^{k-1} - \sum_{\ell=1}^{j-1} v_\ell^k.$$

Recalling that $v_j^k \in H_j$, we obtain that v_j^k is solution to

$$\Delta v_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad v_j^k = 0 \text{ on } \partial \Omega, \quad v_j^k = g_j - \sum_{\ell=j+1}^J v_\ell^{k-1} - \sum_{\ell=1}^{j-1} v_\ell^k \text{ on } \partial O_j. \quad (3.9)$$

The equivalence between (3.9) and (3.5)-(3.6) is proved in the following theorem.

Theorem 3.1. *Consider the sequences $\{v_j^k\}_{k \in \mathbb{N}} \subset H_j$ and $\{d_j^k\}_{k \in \mathbb{N}^+} \subset H_j$ for $j = 1, \dots, J$, and $\{u^k\}_{k \in \mathbb{N}} \subset H$ with $u^0 = \sum_{j=1}^J v_j^0$. Assume that $v_j^k = v_j^{k-1} + d_j^k$ for $k \geq 1$ and $j = 1, \dots, J$. Then the following statements are equivalent:*

- (a) *the d_j^k solve (3.5)-(3.6) and $u^k = u^{k-1} + \sum_{j=1}^J d_j^k = u^0 + \sum_{n=1}^k \sum_{j=1}^J d_j^n$.*
- (b) *the v_j^k solve (3.9) and $u^k = \sum_{j=1}^J v_j^k$.*

Proof. The implication (a) \Rightarrow (b) is proved in the discussion before the theorem. Hence, we need to show that (b) \Rightarrow (a). First, using $v_j^k = v_j^{k-1} + d_j^k$ we get

$$u^k = \sum_{j=1}^J v_j^k = u^0 + \sum_{n=1}^k \sum_{j=1}^J (v_j^n - v_j^{n-1}) = u^0 + \sum_{n=1}^k \sum_{j=1}^J d_j^n = u^{k-1} + \sum_{j=1}^J d_j^k. \quad (3.10)$$

Notice that $d_j^k = v_j^k - v_j^{k-1}$. Clearly, $d_j^k \in H_j$ because $v_j^k, v_j^{k-1} \in H_j$. Hence we have to prove that d_j^k satisfies the transmission condition of (3.5). For $k = 1$, we use that $d_j^1 = v_j^1 - v_j^0$ and the boundary condition in (3.9) for $k = 1$ to write $d_j^1 = v_j^1 - v_j^0 = g_j - \sum_{\ell=1}^{j-1} d_\ell^1 - u^0$. In the case $k > 1$, we write the transmission condition on ∂O_j of (3.9) for k and $k - 1$: $v_j^k = g_j - \sum_{\ell=1}^{j-1} v_\ell^k - \sum_{\ell=j+1}^J v_\ell^{k-1}$ and $v_j^{k-1} = g_j - \sum_{\ell=1}^{j-1} v_\ell^{k-1} - \sum_{\ell=j+1}^J v_\ell^{k-2}$. Subtracting term by term these equations, we get the transmission condition of (3.5). \blacksquare

Using (3.9), we now rewrite Algorithm 1 in terms of v_j^k to obtain Algorithm 2. We next show that this new formulation of the AltMR in terms of the decomposition functions v_j^k allows us to easily obtain a substructured formulation of the AltMR procedure.

3.3. Substructured AltMR as block Gauss-Seidel method

To write the AltMR in substructured form, we recall Lemma 2.1, which plays a very important role in our interpretation of the method of reflections, since it ensures that the unique solution $u \in H$ to (2.3) can be uniquely decomposed as $v_1 + v_2 + \dots + v_J$ with $v_j \in H_j$. For each of these v_j , let us denote by \tilde{g}_j the corresponding Dirichlet trace on ∂O_j so that, because of Definition (2.5), we have

$$\Delta v_j = 0 \text{ in } \Omega \setminus \partial O_j, \quad v_j = 0 \text{ on } \partial \Omega, \quad v_j = \tilde{g}_j \text{ on } \partial O_j. \quad (3.11)$$

Now, we introduce the operator G_j that maps Dirichlet boundary data into functions that are harmonic in $\Omega \setminus \partial O_j$, and such that the solution to (3.11) can be written as $v_j = G_j(\tilde{g}_j)$. The operator G_j can be

Algorithm 2 Alternating Method of Reflections (AltMR)

Require: K (maximum number of iterations), tol (tolerance).

- 1: Set $v_j^0 \in H_j$ for $j = 1, \dots, J$, and $k = 1$.
 - 2: **for** $j = 1:J$ **do**
 - 3: Compute v_j^1 solving problem (3.9).
 - 4: **end for**
 - 5: Compute the approximation $u^1 = \sum_{j=1}^J v_j^1$.
 - 6: **while** $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
 - 7: Update $k = k + 1$.
 - 8: **for** $j = 1:J$ **do**
 - 9: Compute v_j^k solving problem (3.9).
 - 10: **end for**
 - 11: Compute the approximation $u^k = \sum_{j=1}^J v_j^k$.
 - 12: **end while**
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explicitly expressed in terms of Green's representation formulas, and, in particular, as the composition of a single-layer potential integral operator and Dirichlet-to-Neumann operators: for $\bar{g} \in H^{1/2}(\partial O_j)$, we have

$$G_j(\bar{g})(\mathbf{x}) := \int_{\partial O_j} G(\mathbf{x}, \mathbf{y})(\text{DtN}_{j,e}(\bar{g}) + \text{DtN}_{j,i}(\bar{g}))(\mathbf{y}) ds(\mathbf{y}),$$

where $G(\mathbf{x}, \mathbf{y})$ is the Green's function associated to the problem, $\text{DtN}_{j,i} : H^{1/2}(\partial O_j) \rightarrow H^{-1/2}(\partial O_j)$ and $\text{DtN}_{j,e} : H^{1/2}(\partial O_j) \rightarrow H^{-1/2}(\partial O_j)$. These Dirichlet-to-Neumann operators are obtained by first solving the interior and exterior problems for O_j ,

$$\begin{aligned} \Delta v_e &= 0 \text{ in } \Omega \setminus \overline{O_j}, & \Delta v_i &= 0 \text{ in } O_j, \\ v_e &= \bar{g} \text{ on } \partial O_j, & v_i &= \bar{g} \text{ on } \partial O_j, \\ v_e &= 0 \text{ on } \partial \Omega, & & \end{aligned}$$

and then extracting the Neumann trace on ∂O_j of their solutions, that is $\text{DtN}_{j,e}(\bar{g}) := \partial_n v_e(\bar{g})|_{\partial O_j}$ and $\text{DtN}_{j,i}(\bar{g}) := -\partial_n v_i(\bar{g})|_{\partial O_j}$, with $\partial_n = n \cdot \nabla$ the unit outward normal derivative with respect to $\Omega \setminus \overline{O_j}$. In addition, we also need the trace operator $\tau_j : H^1(\Omega) \rightarrow H^{1/2}(\partial O_j)$ such that $\tau_j v$ is the Dirichlet trace of v on ∂O_j , $\tau_j v := v|_{\partial O_j}$.

With this framework, we can rewrite problem (2.2) in integral form: we look for boundary data \tilde{g}_j , for $j = 1, \dots, J$, such that the function $u := \sum_{j=1}^J G_j(\tilde{g}_j)$ satisfies

$$\tau_j u = \tau_j \left(\sum_{\ell=1}^J G_\ell(\tilde{g}_\ell) \right) = g_j, \text{ on } \partial O_j \text{ for } j = 1, \dots, J. \quad (3.12)$$

Equation (3.12) is equivalent to the linear system

$$A \tilde{\mathbf{g}} = \mathbf{g}, \quad (3.13)$$

where

$$A := \begin{bmatrix} I_1 & \tau_1 G_2 & \tau_1 G_3 & \cdots & \tau_1 G_J \\ \tau_2 G_1 & I_2 & \tau_2 G_3 & \cdots & \tau_2 G_J \\ \tau_3 G_1 & \tau_3 G_2 & I_3 & \cdots & \tau_3 G_J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau_J G_1 & \tau_J G_2 & \tau_J G_3 & \cdots & I_J \end{bmatrix}, \quad \tilde{\mathbf{g}} := \begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \\ \vdots \\ \tilde{g}_J \end{bmatrix}, \quad \text{and } \mathbf{g} := \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_J \end{bmatrix}, \quad (3.14)$$

and I_j are identity operators, and we used the fact that $\tau_j G_j = I_j$.

The following theorem shows that the linear system (3.13) is an equivalent formulation of the problem (2.3).

Theorem 3.2. *The system (3.13) is equivalent to (2.3). Moreover, the solution u to (2.3) is uniquely decomposed as $u = \sum_{j=1}^J G_j(\tilde{g}_j)$, where $\tilde{\mathbf{g}} := [\tilde{g}_1 \cdots \tilde{g}_J]^\top$ is the solution to (3.13).*

Proof. The equivalence between (2.3) and (3.13) follows by Lemma 2.1 and the fact that each component v_j of u is uniquely determined by the Dirichlet boundary data \tilde{g}_j on ∂O_j . The second statement is obtained by Lemma 2.1 and the following argument: from (3.12) we have that $\tau_j u = g_j$ on ∂O_j for $j = 1, \dots, J$, and we have that $\Delta u = \Delta \sum_{j=1}^J G_j(\tilde{g}_j) = 0$ on $\Omega \setminus \cup_j \partial O_j$. ■

Remark 3.3. The variables \tilde{g}_j are not necessarily of Dirichlet type: one could formulate the problem with, e.g., Neumann data. In this case the definition of G_j would not require Dirichlet-to-Neumann operators, and $\tau_j G_j$ would be a Neumann-to-Dirichlet operator.

The linear system (3.13) can be solved by classical iterative methods as discussed in the next subsections. For this purpose, we consider the splitting $A = D + L + U$, where $D := \text{diag}(A) = I$,

$$L := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \tau_2 G_1 & 0 & 0 & \cdots & 0 \\ \tau_3 G_1 & \tau_3 G_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tau_J G_1 & \tau_J G_2 & \tau_J G_3 & \cdots & 0 \end{bmatrix}, \quad U := \begin{bmatrix} 0 & \tau_1 G_2 & \tau_1 G_3 & \cdots & \tau_1 G_J \\ 0 & 0 & \tau_2 G_3 & \cdots & \tau_2 G_J \\ 0 & 0 & 0 & \cdots & \tau_3 G_J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This decomposition leads to the classical iterative methods based on the splitting $A = M - N$, and we can write the iterative methods in the standard form $M\tilde{\mathbf{g}}^{k+1} = N\tilde{\mathbf{g}}^k + \mathbf{g}$ and in the difference form $M\boldsymbol{\delta}^{k+1} = N\boldsymbol{\delta}^k$, with $\boldsymbol{\delta}^k := \tilde{\mathbf{g}}^k - \tilde{\mathbf{g}}^{k-1}$. The convergence of this class of methods is related to the iteration operator $G := M^{-1}N$.

Let us now consider the block Gauss-Seidel method for the solution of system (3.13), which is obtained via the splitting $A = M - N$ with $M := D + L$ and $N := -U$,

$$(D + L)\tilde{\mathbf{g}}^{k+1} = -U\tilde{\mathbf{g}}^k + \mathbf{g}, \quad (3.15)$$

and the difference form is given by

$$(D + L)\boldsymbol{\delta}^{k+1} = -U\boldsymbol{\delta}^k. \quad (3.16)$$

where the components of the difference $\boldsymbol{\delta}^k$ are defined by $\delta_j^k := \tilde{g}_j^k - \tilde{g}_j^{k-1}$. Now we show that (3.15) and (3.16) are equivalent to (3.9) and (3.5).

Theorem 3.4. *Assume that $v^0 \in H$ and $\tilde{g}_j^0 = \tau_j v^0$. Then the AltMR methods (3.9) and (3.5) are equivalent to (3.15) and (3.16).*

Proof. Consider the transmission condition of (3.9), that is

$$\tau_j v_j^k = g_j - \sum_{\ell=j+1}^J \tau_j v_\ell^{k-1} - \sum_{\ell=1}^{j-1} \tau_j v_\ell^k, \quad (3.17)$$

for $j = 1, \dots, J$. We denote by $\hat{g}_j^k = \tau_j v_j^k$ and, recalling that v_j^{k-1} solves (3.9), we can use the operator G_j defined above to write $v_j^{k-1} = G_j(\tau_j v_j^{k-1}) = G_j(\hat{g}_j^{k-1})$. Therefore, (3.17) can be equivalently written as

$$\hat{g}_j^k = g_j - \sum_{\ell=1}^{j-1} \tau_j G_j(\hat{g}_\ell^k) - \sum_{\ell=j+1}^J \tau_j G_j(\hat{g}_\ell^{k-1}). \quad (3.18)$$

If $\widehat{g}_j^0 = \widetilde{g}_j^0$, then equation (3.18) is equivalent to the j th equation in (3.15) with $\widehat{g}_j^k = \widetilde{g}_j^k$. In a similar way, using the last equation in (3.5), we obtain that $\delta_j^k = -\sum_{\ell=1}^{j-1} \tau_j G_j(\delta_\ell^k) - \sum_{\ell=j+1}^J \tau_j G_j(\delta_\ell^{k-1})$, which is equivalent to (3.16). \blacksquare

3.4. Analogy with the alternating Schwarz method

We now prove that if $J = 2$, the AltMR is equivalent to the AltSM in the sense of Theorem 3.5. This identification does not hold in general for $J > 2$. To do so, consider problem (2.2), and define for $j = 1, \dots, J$, $\Omega_j := \Omega \setminus \overline{O_j}$, and a smooth enough function $g : \Omega \rightarrow \mathbb{R}$ such that $g|_{\partial O_j} = g_j$, $g|_{\partial \Omega} = 0$, and we assume that there exists a smooth extension of u inside the objects that we denote by $\mathcal{E}_j u$. Then, we set

$$\widetilde{u} := u - g \text{ in } \Omega_j, \quad \widetilde{u} := \mathcal{E}_j u - g \text{ in } O_j.$$

Notice that the extension $\mathcal{E}_j u$ must be smooth enough, e.g. twice differentiable, such that its Laplacian is well defined. Next, we define $f : \Omega \rightarrow \mathbb{R}$ as

$$f := \Delta g \text{ in } \Omega_j, \quad f := \Delta \mathcal{E}_j u + \Delta g \text{ in } \overline{O_j}.$$

Therefore, \widetilde{u} solves the problem

$$\Delta \widetilde{u} = f \text{ in } \Omega \text{ with } \widetilde{u} = 0 \text{ on } \partial \Omega, \tag{3.19}$$

and by construction it satisfies $\widetilde{u} = 0$ on $\bigcup_{j=1}^J \partial O_j$. We can now apply the AltSM to solve (3.19). To do so, we consider the domain decomposition $\Omega = \bigcup_{j=1}^J \Omega_j$, and denote by \widetilde{u}_j^k the approximate solution at the k th iteration on the subdomain Ω_j . Starting with some initial approximation \widetilde{u}_j^0 for all j , the AltSM is defined as

$$\Delta \widetilde{u}_j^k = f \text{ in } \Omega_j, \quad \widetilde{u}_j^k = 0 \text{ on } \partial \Omega, \quad \widetilde{u}_j^k = \sum_{\ell=1}^{j-1} \alpha_{j,\ell} \widetilde{u}_\ell^k + \sum_{\ell=j+1}^J \alpha_{j,\ell} \widetilde{u}_\ell^{k-1} \text{ on } \partial O_j,$$

where the coefficients $\alpha_{j,\ell}$ are non-negative and such that $\sum_{\ell=1}^J \alpha_{j,\ell} = 1$; see, e.g., [17, 51, 15]. This definition implies that the corresponding differences $\widetilde{d}_j^k := \widetilde{u}_j^k - \widetilde{u}_j^{k-1}$ solve

$$\Delta \widetilde{d}_j^k = 0 \text{ in } \Omega_j, \quad \widetilde{d}_j^k = 0 \text{ on } \partial \Omega, \quad \widetilde{d}_j^k = \sum_{\ell=1}^{j-1} \alpha_{j,\ell} \widetilde{d}_\ell^k + \sum_{\ell=j+1}^J \alpha_{j,\ell} \widetilde{d}_\ell^{k-1} \text{ on } \partial O_j. \tag{3.20}$$

Next, we prove in Theorem 3.5 the analogy between the AltMR and the AltSM for $J = 2$, and afterwards we provide a counterexample to show that this analogy does not hold in general for $J > 2$.

Theorem 3.5. *Consider $J = 2$ and assume that $\widetilde{d}_2^1 = d_2^1$. Then for any $k \geq 2$ we have that $\widetilde{d}_2^k = d_2^k$ on $\overline{\Omega}_1$ and $\widetilde{d}_1^k = -d_1^k$ on $\overline{\Omega}_2$, where \widetilde{d}_j^k and d_j^k solve (3.20) and (3.5).*

Proof. Since $J = 2$, we have that $\alpha_{1,2} = \alpha_{2,1} = 1$. We first prove the relation $\widetilde{d}_2^k = d_2^k$ by induction. The result is true for $k = 0$ by assumption. Now, we assume that the relation holds for k and prove that it is true for $k + 1$ as well. Recalling the transmission conditions of (3.20) and (3.5) (for $J = 2$ and $\alpha_{1,2} = \alpha_{2,1} = 1$), it holds that

$$\widetilde{d}_1^{k+1} = \widetilde{d}_2^k = d_2^k = -d_1^{k+1} \text{ on } \partial O_1, \tag{3.21}$$

where we used the induction hypothesis. Equation (3.21), together with the existence of unique solutions to (3.20) and (3.5), implies that $\widetilde{d}_1^{k+1} = -d_1^{k+1}$ on $\overline{\Omega}_1$. Using this equality and the transmission

conditions of (3.20) and (3.5), we have that $\tilde{d}_2^{k+1} = \tilde{d}_1^{k+1} = -d_1^{k+1} = d_2^{k+1}$ on ∂O_2 . Hence the well-posedness of (3.20) and (3.5) implies that $\tilde{d}_2^k = d_2^k$ on $\bar{\Omega}_2$, which is our first claim. The second relation follows directly by the first one together with (3.21). \blacksquare

The equivalence proved in Theorem 3.5 does not hold in general for $J > 2$. In fact, we now show that classical choices of Schwarz methods (like the classical AltSM) does not lead to the result proved in Theorem 3.5. This negative result is suggested by the transmission conditions of (3.20) and (3.5). In fact, if we set $\alpha_{\ell,j} = -1$ for all j, ℓ , the transmission condition of (3.20) coincides with the ones of (3.5). However, this is not possible in a Schwarz method framework, where the hypothesis $\sum_{\ell=1}^J \alpha_{j,\ell} = 1$, with $\alpha_{j,\ell}$ non-negative, is required. Nevertheless, we provide the following example to show that the iterates \tilde{d}_j^k and d_j^k are unrelated for $J > 2$. To do so, consider a domain $\Omega = (0, 1)$ and three holes $O_j = (a_j, b_j)$ for $j = 1, 2, 3$ with $a_j = \frac{2j-1}{7}$ and $b_j = \frac{2j}{7}$. The alternating Schwarz method (3.20) becomes

$$\begin{cases} \Delta \tilde{d}_1^k = 0 \text{ in } \Omega_1, & \tilde{d}_1^k(0) = \tilde{d}_1^k(1) = 0, \\ \tilde{d}_1^k(\tilde{x}) = \alpha_{1,2} \tilde{d}_2^{k-1}(\tilde{x}) + \alpha_{1,3} \tilde{d}_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_1, b_1, \\ \Delta \tilde{d}_2^k = 0 \text{ in } \Omega_2, & \tilde{d}_2^k(0) = \tilde{d}_2^k(1) = 0, \\ \tilde{d}_2^k(\tilde{x}) = \alpha_{2,1} \tilde{d}_1^k(\tilde{x}) + \alpha_{2,3} \tilde{d}_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_2, b_2, \\ \Delta \tilde{d}_3^k = 0 \text{ in } \Omega_3, & \tilde{d}_3^k(0) = \tilde{d}_3^k(1) = 0, \\ \tilde{d}_3^k(\tilde{x}) = \alpha_{3,1} \tilde{d}_1^k(\tilde{x}) + \alpha_{3,2} \tilde{d}_2^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_3, b_3. \end{cases}$$

Notice that by setting $\alpha_{\ell,j} = -1$, the above problems coincide with the AltMR (3.5). The general solutions to these three problems are

$$\tilde{d}_1^k(x) = \begin{cases} \frac{A_1^k x}{a_1} & x \in [0, a_1], \\ \frac{B_1^k(1-x)}{1-b_1} & x \in [b_1, 1], \end{cases} \quad \tilde{d}_2^k(x) = \begin{cases} \frac{A_2^k x}{a_2} & x \in [0, a_2], \\ \frac{B_2^k(1-x)}{1-b_2} & x \in [b_2, 1], \end{cases} \quad \tilde{d}_3^k(x) = \begin{cases} \frac{A_3^k x}{a_3} & x \in [0, a_3], \\ \frac{B_3^k(1-x)}{1-b_3} & x \in [b_3, 1], \end{cases}$$

where A_j^k and B_j^k are constants depending on the transmission conditions. Defining $\mathbf{v}^k := [A_1^k, B_1^k, A_2^k, B_2^k, A_3^k, B_3^k]^\top$ and using the transmission conditions, we obtain the iteration relation

$$(I + \tilde{L})\mathbf{v}^k = -\tilde{U}\mathbf{v}^{k-1},$$

where

$$\tilde{L} = - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_{2,1}(1-a_2)}{(1-b_1)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_{2,1}(1-b_2)}{(1-b_1)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_{3,1}(1-a_3)}{(1-b_1)} & 0 & \frac{\alpha_{3,2}(1-a_3)}{(1-b_2)} & 0 & 0 \\ 0 & \frac{\alpha_{3,1}(1-b_3)}{(1-b_1)} & 0 & \frac{\alpha_{3,2}(1-b_3)}{(1-b_2)} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{U} = - \begin{bmatrix} 0 & 0 & \frac{\alpha_{1,2}a_1}{a_2} & 0 & \frac{\alpha_{1,3}a_1}{a_3} & 0 \\ 0 & 0 & \frac{\alpha_{1,2}b_1}{a_2} & 0 & \frac{\alpha_{1,3}b_1}{a_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_{2,3}a_2}{a_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha_{2,3}b_2}{a_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us define the matrix $G = -(I_6 + \tilde{L})^{-1}\tilde{U}$, with I_6 the 6×6 identity. The iteration relation above then reads $\mathbf{v}^k = G\mathbf{v}^{k-1}$, where the matrix G depends on the weights $\alpha_{j,k}$. Notice that the same relation (with appropriately chosen weights in G) can be obtained for the AltMR. We consider three different cases. The first case is $\alpha_{1,2} = \alpha_{2,3} = \alpha_{3,1} = 0$ and $\alpha_{1,3} = \alpha_{2,1} = \alpha_{3,2} = 1$, which corresponds to the classical AltSM, the second case is $\alpha_{j,k} = \frac{1}{2}$ for any j, k , which leads to a weighted AltSM, and the third case $\alpha_{j,k} = -1$ for any j, k , which corresponds to the AltMR. Computing explicitly the iteration

matrix G is these cases, we get

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{25} & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & \frac{1}{6} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{15} & 0 & \frac{19}{30} & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{29}{30} & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{150}{29} & 0 \\ 0 & 0 & \frac{1}{20} & 0 & \frac{29}{300} & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 0 & -\frac{1}{33} & 0 & -\frac{1}{25} & 0 \\ 0 & 0 & -\frac{1}{33} & 0 & -\frac{1}{25} & 0 \\ 0 & 0 & \frac{8}{15} & 0 & -\frac{14}{25} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & -\frac{14}{25} & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{15} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{15} & 0 \end{bmatrix}.$$

Their spectra are

$$\sigma(G_1) = \left\{ \frac{4}{25}, 0 \right\}, \quad \sigma(G_2) = \left\{ \frac{-(3\sqrt{389} - 49)}{300}, \frac{49 + 3\sqrt{389}}{300}, 0 \right\}, \quad \sigma(G_3) = \left\{ \frac{8}{15}, 0 \right\},$$

and we thus obtain for the spectral radii

$$\rho(G_1) = \frac{4}{25} = 0.16, \quad \rho(G_2) = \frac{49 + 3\sqrt{389}}{300} \approx 0.36, \quad \rho(G_3) = \frac{8}{15} \approx 0.53.$$

This shows that the iterations generated by G_1 and G_2 corresponding to alternating Schwarz methods must be different from the iterations generated by G_3 corresponding to the AltMR, and we have thus shown that there is in general no relation between the AltMR and the AltSM for $J > 2$.

The reader may ask himself how the iteration matrices look like for $J = 2$. To illustrate this, we consider a domain $\Omega = (0, 1)$ with two holes $O_j = (a_j, b_j)$ for $j = 1, 2$ with $a_j = \frac{2j-1}{5}$ and $b_j = \frac{2j}{5}$, and recall that $\alpha_{1,2} = \alpha_{2,1} = 1$. The same arguments as above allow us to obtain the iteration matrices

$$G_{\text{AltSM}} = \begin{bmatrix} 0 & 0 & \frac{1}{33} & 0 \\ 0 & 0 & \frac{4}{33} & 0 \\ 0 & 0 & \frac{4}{33} & 0 \\ 0 & 0 & \frac{9}{9} & 0 \end{bmatrix}, \quad G_{\text{AltMR}} = \begin{bmatrix} 0 & 0 & -\frac{1}{33} & 0 \\ 0 & 0 & -\frac{4}{33} & 0 \\ 0 & 0 & \frac{4}{33} & 0 \\ 0 & 0 & \frac{9}{9} & 0 \end{bmatrix},$$

whose spectra coincide and are given by $\sigma(G_{\text{AltMR}}) = \sigma(G_{\text{PSM}}) = \left\{ \frac{4}{9}, 0 \right\}$. It is clear that the two matrices generate similar iterates. Notice also the negative signs in G_{AltMR} that produce d_1^k with opposite sign to \tilde{d}_1^k , in agreement with Theorem 3.5.

3.5. Convergence analysis for $J = 2$ objects

Consider problem (2.2) for $J = 2$:

$$\Delta u = 0 \text{ in } \Omega \setminus (\partial O_1 \cup \partial O_2), \quad u = 0 \text{ on } \partial \Omega, \quad u = g_1 \text{ on } \partial O_1, \quad u = g_2 \text{ on } \partial O_2. \quad (3.22)$$

The AltMR (3.9) for the solution of (3.22) is

$$\begin{aligned} \Delta v_1^k &= 0 \text{ in } \Omega \setminus \partial O_1, \quad v_1^k = 0 \text{ on } \partial \Omega, & \Delta v_2^k &= 0 \text{ in } \Omega \setminus \partial O_2, \quad v_2^k = 0 \text{ on } \partial \Omega, \\ v_1^k &= g_1 - v_2^{k-1} \text{ on } \partial O_1, & v_2^k &= g_2 - v_1^k \text{ on } \partial O_2. \end{aligned} \quad (3.23)$$

Let us define the error at the k th iteration by $e_j^k := v_j - v_j^k$ for $j = 1, \dots, J$. In terms of the errors, the AltMR (3.23) reads

$$\begin{aligned} \Delta e_1^k &= 0 \text{ in } \Omega \setminus \partial O_1, \quad e_1^k = 0 \text{ on } \partial \Omega, & \Delta e_2^k &= 0 \text{ in } \Omega \setminus \partial O_2, \quad e_2^k = 0 \text{ on } \partial \Omega, \\ e_1^k &= -e_2^{k-1} \text{ on } \partial O_1, & e_2^k &= -e_1^k \text{ on } \partial O_2. \end{aligned} \quad (3.24)$$

We can now prove the following theorem using similar techniques considered in [9].

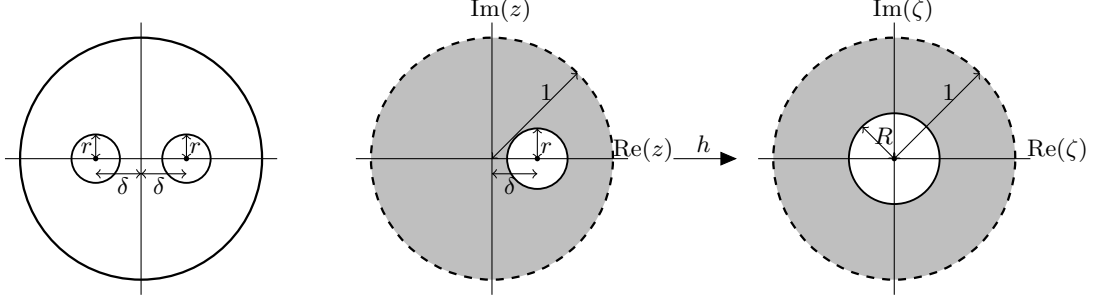


FIGURE 3.1. Left (white disc): A unit disc Ω with two objects O_1 and O_2 , which are two discs having the same radius denoted by r . The distance between the centers of O_1 and O_2 and the center of Ω is denoted by δ . Right (gray discs): Möbius transformation of the unit disc Ω (dashed line) with a hole O_1 (solid line) into an annulus.

Theorem 3.6. *The AltMR for the solution to problem (3.22) converges geometrically, in the sense that*

$$\max_{\overline{\Omega}} |e_j^k| \leq \rho_{\text{AltMR}}^k \max_{\overline{\Omega}} |e_j^0|,$$

for $j = 1, 2$, where $\rho_{\text{AltMR}} = (\max_{\partial O_1} w_2)(\max_{\partial O_2} w_1) < 1$, and w_j solves

$$\Delta w_j = 0 \text{ in } \Omega \setminus \partial O_j, \quad w_j = 0 \text{ on } \partial \Omega, \quad w_j = 1 \text{ on } \partial O_j. \quad (3.25)$$

Proof. The functions w_j , for $j = 1, 2$ which are solutions to (3.25), satisfy because of the maximum principle that $w_j \geq 0$ and $|e_j^k| \leq w_j \max_{\partial O_j} |e_j^k|$. Now, since e_1^k and e_2^k are harmonic functions and solve (3.24), the maximum principle allows us to write

$$\begin{aligned} \max_{\overline{\Omega}} |e_1^k| &= \max_{\partial O_1} |e_1^k| = \max_{\partial O_1} |e_2^{k-1}| \leq \max_{\partial O_1} w_2 \max_{\partial O_2} |e_2^{k-1}| = \max_{\partial O_1} w_2 \max_{\overline{\Omega}} |e_2^{k-1}|, \\ \max_{\overline{\Omega}} |e_2^{k-1}| &= \max_{\partial O_2} |e_2^{k-1}| = \max_{\partial O_2} |e_1^{k-1}| \leq \max_{\partial O_2} w_1 \max_{\partial O_1} |e_1^{k-1}| = \max_{\partial O_2} w_1 \max_{\overline{\Omega}} |e_1^{k-1}|. \end{aligned}$$

By combining these two estimates we get

$$\max_{\overline{\Omega}} |e_1^k| \leq \max_{\partial O_1} w_2 \max_{\overline{\Omega}} |e_2^{k-1}| \leq \max_{\partial O_1} w_2 \max_{\partial O_2} w_1 \max_{\overline{\Omega}} |e_1^{k-1}| = \rho_{\text{AltMR}} \max_{\overline{\Omega}} |e_1^{k-1}|.$$

Since w_1 and w_2 are harmonic functions in $\Omega \setminus \partial O_1$ and $\Omega \setminus \partial O_2$, Assumption (2.1) and the maximum principle imply that $\max_{\partial O_1} w_2 < 1$ and $\max_{\partial O_2} w_1 < 1$. Hence $\rho_{\text{AltMR}} < 1$ and our proof is complete. \blacksquare

This theorem allows us to compute explicitly the contraction factor ρ_{AltMR} as a function of the geometry of the domain. For example, consider a domain whose geometry is shown in Figure 3.1 (left), where $\Omega \subset \mathbb{R}^2$ is the unit disc, the objects O_1 and O_2 are two discs whose centers are aligned on a straight line passing through the center of $\Omega \subset \mathbb{R}^2$. The two discs have the same radius r and the distance between the center of each of them and the center of Ω is denoted by δ . We can prove the following result.

Corollary 3.7. *Consider the problem (3.22) defined on a domain whose geometry is depicted in Figure 3.1 (left). We have that*

$$\rho_{\text{AltMR}}(r, \delta) = \left(\frac{\log \left(\frac{(-a-\delta+r)^2}{(a(r-\delta)-1)^2} \right)}{\log \left(\frac{(-a+\delta-r)^2}{(a(\delta-r)-1)^2} \right)} \right)^2,$$

where

$$a = \frac{1 + \delta^2 - r^2 - \sqrt{r^4 + (-2 - 2\delta^2)r^2 + \delta^4 - 2\delta^2 + 1}}{2\delta}. \quad (3.26)$$

Proof. To prove the result, we first need to solve problems (3.25) and compute the w_j . To do so, we recall the Möbius transformation $h : \mathbb{C} \rightarrow \mathbb{C}$:

$$h(z) := \frac{z - a}{\bar{a}z - 1}, \quad (3.27)$$

which maps the unit circle into itself and circles into circles. In particular, by imposing the conditions $h(\delta - r) = R$ and $h(\delta + r) = -R$, one obtains a (real) coefficient a , as in (3.26), such that h maps $\Omega \setminus \overline{O_j}$ into an annulus; see Figure 3.1 (right). We consider now the problem

$$\Delta \tilde{w}_1 = 0 \text{ in } h(\Omega) \setminus h(\partial O_1), \quad \tilde{w}_1 = 0 \text{ on } h(\partial \Omega), \quad \tilde{w}_1 = 1 \text{ on } h(\partial O_1). \quad (3.28)$$

Since (3.28) is radially symmetric, a separation of variables argument allows us to compute its solution,

$$\tilde{w}_1(\zeta) = \frac{\log(\sqrt{\operatorname{Re}(\zeta)^2 + \operatorname{Im}(\zeta)^2})}{\log(R)}.$$

By transforming back from ζ to z , we obtain

$$w_1(\operatorname{Re}(z), \operatorname{Im}(z)) = \tilde{w}_1(h(z)) = \frac{\log(|h(z)|)}{\log(R)} = \frac{\log(|h(z)|^2)}{\log(|R|^2)} = \frac{\log\left(\left|\frac{z-a}{az-1}\right|^2\right)}{\log\left(\left|\frac{\delta-r-a}{a(\delta-r)-1}\right|^2\right)},$$

which solves (3.25) for $j = 1$ since h is a conformal mapping. A further simplification leads to

$$w_1(x, y) = \frac{\log\left(\frac{(x-a)^2 + y^2}{(ax-1)^2 + (ay)^2}\right)}{\log\left(\frac{(\delta-r-a)^2}{(a(\delta-r)-1)^2}\right)}.$$

Now, since \tilde{w}_1 is symmetric with respect to the $\operatorname{Re}(\zeta)$ -axis and decays monotonically in any radial direction, the same holds for w_1 . Hence, the maximum of w_1 along ∂O_2 is attained at $(x, y) = (-\delta + r, 0)$ and has the value

$$\max_{\partial O_2} w_1 = w_1(-\delta + r, 0) = \frac{\log\left(\frac{(-\delta+r-a)^2}{(a(-\delta+r)-1)^2}\right)}{\log\left(\frac{(\delta-r-a)^2}{(a(\delta-r)-1)^2}\right)}.$$

Since the solution w_2 to problem (3.25) for $j = 2$ can be obtained by rotating w_1 by π around the origin, it holds that

$$\max_{\partial O_1} w_2 = w_2(\delta - r, 0) = w_1(-\delta + r, 0) = \max_{\partial O_2} w_1.$$

Recalling from Theorem 3.6 that $\rho_{\text{AltMR}} = (\max_{\partial O_1} w_2)(\max_{\partial O_2} w_1)$, the claim follows. \blacksquare

4. The parallel method of reflections

The parallel method of reflections (PMR) was introduced by Golusin in 1934 [24] and formally recalled by Traytak [52] for the Laplace equation. Ichiki and Brady [31] present exactly the parallel method of Golusin, and they state: ‘‘It is easy to extend this procedure to the N body problem by superposing distances by other particles’’. In practice, the parallel version is obtained by replacing in the right-hand side of the boundary condition (3.4) the differences at the iteration k with the corresponding ones at the iteration $k - 1$. Hence, problem (3.5) becomes

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial \Omega, \quad d_j^k = - \sum_{\ell=1, \ell \neq j}^J d_\ell^{k-1} \text{ on } \partial O_j. \quad (4.1)$$

The sequences $\{d_j^k\}_{k \in \mathbb{N}^+}$ are initialized by solving for each $j = 1, \dots, J$ the problem

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^1 = 0 \text{ on } \partial \Omega, \quad d_j^1 = g_j - u^0 \text{ on } \partial O_j, \quad (4.2)$$

for some $u^0 \in H$, and the approximate solution at the k th iteration is defined by

$$u^k = u^{k-1} + \sum_{j=1}^J d_j^k. \quad (4.3)$$

The PMR (4.1)-(4.2), as presented in [38], leads to Algorithm 3.

Algorithm 3 Parallel Method of Reflection (PMR)

Require: K (maximum number of iterations), tol (tolerance).

- 1: Set $u^0 \in H$ and $k = 1$.
 - 2: **for** $j = 1:J$ (this loop is executed in parallel) **do**
 - 3: Compute d_j^1 solving problem (4.2).
 - 4: **end for**
 - 5: Compute the approximation $u^1 = \sum_{j=1}^J d_j^1$.
 - 6: **while** $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
 - 7: Update $k = k + 1$.
 - 8: **for** $j = 2:J$ (this loop is executed in parallel) **do**
 - 9: Compute d_j^k solving problem (4.1).
 - 10: **end for**
 - 11: Compute the approximation $u^k = u^{k-1} + \sum_{j=1}^J d_j^k$.
 - 12: **end while**
-

Now, as in Section 3, we introduce the variable v_j^k defined in (3.7), and we write the PMR in terms of v_j^k . To do so, we first use the transmission condition of (4.1) and (4.3) to compute on ∂O_j that $d_j^k = -\sum_{\ell=1, \ell \neq j}^J d_\ell^{k-1} = d_j^{k-1} - (u^{k-1} - u^{k-2})$, which implies that $d_j^k + u^{k-1} = d_j^{k-1} + u^{k-2} = \dots = d_j^1 + u^0$. Recalling that $d_j^1 = g_j - u^0$ on ∂O_j , we obtain $d_j^k = g_j - u^{k-1}$ on ∂O_j . Therefore, we have that (4.1) is equivalent to

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial \Omega, \quad d_j^k = g_j - u^{k-1} \text{ on } \partial O_j, \quad (4.4)$$

which is the parallel version of (3.3). Now, starting with the definition (3.7) and using $d_j^k = g_j - u^{k-1}$ on ∂O_j and (3.8), we compute on ∂O_j that $v_j^k = v_j^{k-1} + d_j^k = v_j^{k-1} + g_j - u^{k-1} = g_j - \sum_{\ell=1, \ell \neq j}^J v_\ell^{k-1}$. Recalling that $v_j^k \in H_j$, we conclude that v_j^k solves

$$\Delta v_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad v_j^k = 0 \text{ on } \partial \Omega, \quad v_j^k = g_j - \sum_{\ell=j+1}^J v_\ell^{k-1} - \sum_{\ell=1}^{j-1} v_\ell^{k-1} \text{ on } \partial O_j. \quad (4.5)$$

The equivalence between (4.5) and (4.1)-(4.2) is given in the following theorem that can be proved similarly as Theorem 3.1.

Theorem 4.1. *Consider the sequences $\{v_j^k\}_{k \in \mathbb{N}} \subset H_j$ and $\{d_j^k\}_{k \in \mathbb{N}^+} \subset H_j$ for $j = 1, \dots, J$, and $\{u^k\}_{k \in \mathbb{N}} \subset H$ with $u^0 = \sum_{j=1}^J v_j^0$. Assume that $v_j^k = v_j^{k-1} + d_j^k$ for $k \geq 1$ and $j = 1, \dots, J$. Then the following statements are equivalent:*

- (a) *the d_j^k solve (4.1)-(4.2) with $u^k = u^{k-1} + \sum_{j=1}^J d_j^k = u^0 + \sum_{n=1}^k \sum_{j=1}^J d_j^n$.*
- (b) *the v_j^k solve (4.5) with $u^k = \sum_{j=1}^J v_j^k$.*

The PMR algorithm in terms of v_j^k is given in Algorithm 4.

Algorithm 4 Parallel Method of Reflection (PMR)

Require: K (maximum number of iteration), tol (tolerance).

- 1: Set $v_j^0 \in H_j$ for $j = 1, \dots, J$, and $k = 1$.
 - 2: **for** $j = 1:J$ (this loop is executed in parallel) **do**
 - 3: Compute v_j^1 solving problem (4.5).
 - 4: **end for**
 - 5: Compute the approximation $u^1 = \sum_{j=1}^J v_j^1$.
 - 6: **while** $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
 - 7: Update $k = k + 1$.
 - 8: **for** $j = 1:J$ (this loop is executed in parallel) **do**
 - 9: Compute v_j^k solving problem (4.5).
 - 10: **end for**
 - 11: Compute the approximation $u^k = \sum_{j=1}^J v_j^k$.
 - 12: **end while**
-

4.1. Substructured PMR as a block Jacobi method

Ichiki and Brady [31, p. 351] already mention, without motivation or any rigorous argument, that “this iterative method (the PMR) is equivalent to the block Jacobi method”. To see this, consider the operator A defined in Section 3.3 and the decomposition $A = M - N$ with $M := D$ and $N := -(L + U)$. The block Jacobi method in the standard form is given by

$$D\tilde{\mathbf{g}}^{k+1} = -(L + U)\tilde{\mathbf{g}}^k + \mathbf{g}, \quad (4.6)$$

and in the difference form it is

$$D\boldsymbol{\delta}^{k+1} = -(L + U)\boldsymbol{\delta}^k, \quad (4.7)$$

where $\boldsymbol{\delta}^k := \tilde{\mathbf{g}}^k - \tilde{\mathbf{g}}^{k-1}$. The next theorem shows that (4.6) and (4.7) are equivalent to the PMR.

Theorem 4.2. *Assume that $v^0 \in H$ and $\tilde{g}_j^0 = \tau_j v^0$. Then the PMR forms (4.5) and (4.4) are equivalent to (4.6) and (4.7).*

Proof. We proceed as in Section 3.3 and consider the transmission condition of (4.5), that is

$$\tau_j v_j^k = g_j - \sum_{\ell=j+1}^J \tau_j v_\ell^{k-1} - \sum_{\ell=1}^{j-1} \tau_j v_\ell^{k-1}. \quad (4.8)$$

We define $\hat{g}_j^k := \tau_j v_j^k$ and recall that v_j^{k-1} can be written as $v_j^{k-1} = G_j(\tau_j v_j^{k-1}) = G_j(\hat{g}_j^{k-1})$. Hence, (4.8) becomes

$$\hat{g}_j^k = g_j - \sum_{\ell=j+1}^J \tau_j G_j(\hat{g}_\ell^{k-1}) - \sum_{\ell=1}^{j-1} \tau_j G_j(\hat{g}_\ell^{k-1}),$$

which is equivalent to (4.6) if $\hat{g}_j^0 = \tilde{g}_j^0$. The same arguments can be used to show that (4.1) is equivalent to (4.7). ■

4.2. Analogies with Schwarz methods

Similar to Section 3.4, we show now that if $J = 2$, then the PMR is the analog of the classical parallel Schwarz method (PSM), and if $J > 2$, the PMR and the PSM define different iterates. To do so, we

recall (3.19) and apply the PSM,

$$\Delta \tilde{u}_j^k = f \text{ in } \Omega_j, \quad \tilde{u}_j^k = 0 \text{ on } \partial\Omega, \quad \tilde{u}_j^k = \sum_{\ell=1, \ell \neq j}^J \alpha_{j,\ell} \tilde{u}_\ell^{k-1} \text{ on } \partial O_j, \quad (4.9)$$

where the coefficients $\alpha_{j,\ell}$ are non-negative and such that $\sum_\ell \alpha_{j,\ell} = 1$. This definition implies that the corresponding differences $\tilde{d}_j^k := u_j^k - u_j^{k-1}$ solve

$$\Delta \tilde{d}_j^k = 0 \text{ in } \Omega_j, \quad \tilde{d}_j^k = 0 \text{ on } \partial\Omega, \quad \tilde{d}_j^k = \sum_{\ell=1, \ell \neq j}^J \alpha_{j,\ell} \tilde{d}_\ell^{k-1} \text{ on } \partial O_j. \quad (4.10)$$

Theorem 4.3. *Consider $J = 2$ and assume that $\tilde{d}_2^1 = d_2^1$. Then, for any $k \geq 1$ we have that $\tilde{d}_1^k = (-1)^k d_1^k$ on $\bar{\Omega}_1$ and $\tilde{d}_2^k = (-1)^k d_2^k$ on $\bar{\Omega}_2$, where \tilde{d}_j^k and d_j^k solve (4.10) and (4.5).*

Proof. Since $J = 2$, we have that $\alpha_{1,2} = \alpha_{2,1} = 1$. We only prove the first relation $\tilde{d}_1^k = (-1)^k d_1^k$, since the second follows by the same arguments. We proceed by induction. The statement is true for $k = 1$ by assumption. Now, we assume that $\tilde{d}_1^k = (-1)^k d_1^k$ holds and we show that it remains true for $k + 1$. Recalling the transmission condition of (4.10) and (4.5), we have

$$d_1^{k+1} = -d_2^k = -(-1)^k \tilde{d}_2^k = (-1)^{k+1} \tilde{d}_2^k = (-1)^k \tilde{d}_1^{k+1} \text{ on } \partial O_1,$$

where we used the induction hypothesis. Since (4.10) and (4.5) admits unique solutions, we have that $d_1^{k+1} = (-1)^k \tilde{d}_1^{k+1}$ in $\bar{\Omega}_1$, which is our claim. \blacksquare

Now, to show that for $J > 2$ the analogy proved in Theorem 4.3 is not in general true, we consider the same example provided in Section 3.4 and use the same notation. In this case, the PSM is

$$\begin{cases} \Delta \tilde{d}_1^k = 0 \text{ in } \Omega_1, & \tilde{d}_1^k(0) = \tilde{d}_1^k(1) = 0, \\ \tilde{d}_1^k(\tilde{x}) = \alpha_{1,2} \tilde{d}_2^{k-1}(\tilde{x}) + \alpha_{1,3} \tilde{d}_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_1, b_1, \\ \Delta \tilde{d}_2^k = 0 \text{ in } \Omega_2, & \tilde{d}_2^k(0) = \tilde{d}_2^k(1) = 0, \\ \tilde{d}_2^k(\tilde{x}) = \alpha_{2,1} \tilde{d}_1^{k-1}(\tilde{x}) + \alpha_{2,3} \tilde{d}_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_2, b_2, \\ \Delta \tilde{d}_3^k = 0 \text{ in } \Omega_3, & \tilde{d}_3^k(0) = \tilde{d}_3^k(1) = 0, \\ \tilde{d}_3^k(\tilde{x}) = \alpha_{3,1} \tilde{d}_1^{k-1}(\tilde{x}) + \alpha_{3,2} \tilde{d}_2^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_3, b_3. \end{cases}$$

The corresponding iteration in terms of the constants A_j^k and B_j^k is given by $\mathbf{v}^k = G \mathbf{v}^{k-1}$, where $G := -(\tilde{L} + \tilde{U})$, with G depending on the weights $\alpha_{\ell,j}$. Notice that the same relation (with appropriately chosen weights in G) can be obtained for the PMR. We consider three different cases. The first case is $\alpha_{1,2} = \alpha_{2,3} = \alpha_{3,1} = 0$ and $\alpha_{1,3} = \alpha_{2,1} = \alpha_{3,2} = 1$, which corresponds to the classical PSM, the second case is $\alpha_{j,k} = \frac{1}{2}$ for any j, k , which leads to a weighted PSM, and the third case $\alpha_{j,k} = -1$ for any j, k , which corresponds to the PMR. Computing explicitly the iteration matrix G in these cases, we get

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 & \frac{1}{6} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{10} & 0 \\ 0 & \frac{3}{10} & 0 & 0 & \frac{2}{5} & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{10} & 0 & \frac{1}{6} & 0 & 0 \end{bmatrix}, \quad G_3 = - \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix},$$

where G_1 , G_2 , and G_3 correspond to the three cases. Their spectra are

$$\begin{aligned}\sigma(G_1) &= \left\{ \frac{i\sqrt{3}4^{1/3} - 4^{1/3}}{2 \cdot 25^{1/3}}, -\frac{4^{1/3} + i\sqrt{3}4^{1/3}}{2 \cdot 25^{1/3}}, \frac{4^{1/3}}{25^{1/3}}, 0 \right\}, \\ \sigma(G_2) &= \left\{ -\frac{3 + \sqrt{39}}{30}, \frac{\sqrt{39} - 3}{30}, -\frac{\sqrt{219} - 3}{30}, \frac{\sqrt{219} + 3}{30}, 0 \right\}, \\ \sigma(G_3) &= \left\{ \frac{3 + \sqrt{39}}{15}, -\frac{\sqrt{39} - 3}{15}, \frac{\sqrt{219} - 3}{15}, -\frac{\sqrt{219} + 3}{15}, 0 \right\},\end{aligned}$$

and the corresponding spectral radii are

$$\rho(G_1) = \frac{4^{1/3}}{25^{1/3}} \approx 0.54, \quad \rho(G_2) = \frac{\sqrt{219} + 3}{30} \approx 0.59, \quad \rho(G_3) = \frac{\sqrt{219} + 3}{15} \approx 1.18.$$

We see again that the iterations generated by G_1 and G_2 corresponding to parallel Schwarz methods are different from the iterations generated by G_3 corresponding to the PMR, and we have therefore shown that there is no relation in general between PMR and PSM for $J > 2$.

4.3. Convergence analysis for $J = 2$ objects

Next, we prove for the PSM the same convergence result obtained for the AltMR in Theorem 3.6. The PMR for the errors e_1^k and e_2^k is

$$\begin{aligned}\Delta e_1^k &= 0 \text{ in } \Omega \setminus \partial O_1, \quad e_1^k = 0 \text{ on } \partial \Omega, & \Delta e_2^k &= 0 \text{ in } \Omega \setminus \partial O_2, \quad e_2^k = 0 \text{ on } \partial \Omega, \\ e_1^k &= -e_2^{k-1} \text{ on } \partial O_1, & e_2^k &= -e_1^{k-1} \text{ on } \partial O_2.\end{aligned}\tag{4.11}$$

Theorem 4.4. *The PMR for the solution to problem (3.22) converges geometrically, in the sense that $\max_{\overline{\Omega}} |e_j^{2k}| \leq \rho_{\text{PMR}}^{2k} \max_{\overline{\Omega}} |e_j^0|$, for $j = 1, 2$, where $\rho_{\text{PMR}} = \sqrt{\rho_{\text{AltMR}}}$ and ρ_{AltMR} is given in Corollary 3.7.*

Proof. As in Corollary 3.7, the functions w_j solving (3.25) satisfy by the maximum principle that $w_j \geq 0$ and $|e_j^k| \leq w_j \max_{\partial O_j} |e_j^k|$. Now, since e_1^k and e_2^k are harmonic functions and solve (3.24), the maximum principle allows us to write

$$\begin{aligned}\max_{\overline{\Omega}} |e_1^k| &= \max_{\partial O_1} |e_1^k| = \max_{\partial O_1} |e_2^{k-1}| \leq \max_{\partial O_1} w_2 \max_{\partial O_2} |e_2^{k-1}| = \max_{\partial O_1} w_2 \max_{\overline{\Omega}} |e_2^{k-1}|, \\ \max_{\overline{\Omega}} |e_2^{k-1}| &= \max_{\partial O_2} |e_2^{k-1}| = \max_{\partial O_2} |e_1^{k-2}| \leq \max_{\partial O_2} w_1 \max_{\partial O_1} |e_1^{k-2}| = \max_{\partial O_2} w_1 \max_{\overline{\Omega}} |e_1^{k-2}|.\end{aligned}$$

By combining these two estimates we get

$$\max_{\overline{\Omega}} |e_1^k| \leq \max_{\partial O_1} w_2 \max_{\overline{\Omega}} |e_2^{k-1}| \leq \max_{\partial O_1} w_2 \max_{\partial O_2} w_1 \max_{\overline{\Omega}} |e_1^{k-2}| = \rho_{\text{AltMR}} \max_{\overline{\Omega}} |e_1^{k-2}|.$$

Using this inequality recursively leads to $\max_{\overline{\Omega}} |e_1^{2k}| \leq \rho_{\text{AltMR}}^{2k} \max_{\overline{\Omega}} |e_1^0|$, which is the claim for $j = 1$. The same arguments allow us to prove this result also for $j = 2$. \blacksquare

Notice that Theorem 3.6 and Theorem 4.4 are proved using maximum principle arguments that are classical techniques for proving convergence of classical Schwarz methods; see, e.g., [40, 9]. Notice also that the estimated contraction factors for AltMR and PMR are one the square of the other. This resembles classical results for Schwarz methods; see, e.g., [16, 17] and references therein.

5. Methods of reflections with relaxation

We now present further forms of MR, and prove that they are equivalent to classical stationary iterative methods with relaxation.

5.1. The averaged parallel method of reflection

Golusin already said in [24] that the parallel method of reflections always converges for $J = 2$ objects, but not for $J > 2$, “but by modifying (3.12) (or (4.6)-(4.7)) appropriately one could increase the area of applicability of the preceding result”. This fact has been mentioned in several publications; see, e.g., [24, 31]. In order to improve the convergence behavior of the PMR, Laurent et al. have recently proposed in [38] a modified version that is obtained (as mentioned by the authors) by averaging the different components d_j^k . To do so, the problem (4.1) is modified by adding a weight,

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial\Omega, \quad d_j^k = \left(1 - \frac{1}{J}\right)d_j^{k-1} - \frac{1}{J} \sum_{\ell=1, \ell \neq j}^J d_\ell^{k-1} \text{ on } \partial O_j, \quad (5.1)$$

for $j = 1, \dots, J$, with the initialization problems

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^1 = 0 \text{ on } \partial\Omega, \quad d_j^1 = g_j - u^0 \text{ on } \partial O_j, \quad (5.2)$$

for $j = 1, \dots, J$ and $u^0 \in H$. The approximate solution u^k can be obtained from u^{k-1} by

$$u^k = u^{k-1} + \frac{1}{J} \sum_{j=1}^J d_j^k, \quad (5.3)$$

assuming that $u^0 = 0$. This new formulation of the method, that we call averaged parallel method of reflections (APMR), is proved to be always convergent in [38]. Now, we want to formulate this new version in terms of v_j^k . To do so, we proceed in a similar way as in Section 3.2 and define the variable v_j^k as $v_j^k := v_j^{k-1} + \frac{1}{J}d_j^k$ with $v_j^0 \in H_j$ (compare with (3.7)). We then obtain that v_j^k solves the problem

$$\Delta v_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad v_j^k = 0 \text{ on } \partial\Omega, \quad v_j^k = \left(1 - \frac{1}{J}\right)v_j^{k-1} + \frac{1}{J}g_j - \frac{1}{J} \sum_{\ell=1, \ell \neq j}^J v_\ell^{k-1} \text{ on } \partial O_j. \quad (5.4)$$

The equivalence between (5.4) and (5.1)-(5.2) is proved in the following theorem, whose proof is similar to the one of Theorem 3.1.

Theorem 5.1. *Consider the sequences $\{v_j^k\}_{k \in \mathbb{N}} \subset H_j$ and $\{d_j^k\}_{k \in \mathbb{N}^+} \subset H_j$ for $j = 1, \dots, J$, and $\{u^k\}_{k \in \mathbb{N}} \subset H$ with $u^0 = \sum_{j=1}^J v_j^0$. Assume that $v_j^k = v_j^{k-1} + \frac{1}{J}d_j^k$ for $k \geq 1$ and $j = 1, \dots, J$. Then the following statements are equivalent:*

- (a) *the d_j^k solve (5.1)-(5.2) with $u^k = u^{k-1} + \frac{1}{J} \sum_{j=1}^J d_j^k = u^0 + \frac{1}{J} \sum_{n=1}^k \sum_{j=1}^J d_j^n$.*
- (b) *the v_j^k solve (5.4) with $u^k = \sum_{j=1}^J v_j^k$.*

The block Jacobi method (4.6) we have presented earlier might not converge if the off-diagonal blocks are too heavy, i.e. the objects are too close (in agreement with the results of Golusin [24] and Traytak [52]); see also [11, Section 3] for detailed one-dimensional examples. However, one can consider a relaxation,

$$\tilde{\mathbf{g}}^k = (1 - \omega)\tilde{\mathbf{g}}^{k-1} + \omega D^{-1}[-(L + U)\tilde{\mathbf{g}}^{k-1} + \mathbf{g}], \quad (5.5)$$

where the parameter ω has to be chosen in a proper way. This is the relaxed block Jacobi method, and we have the following result.

Theorem 5.2. *We have the following equivalences:*

- *The relaxed block Jacobi method (5.5) is equivalent to the damped block Jacobi method, that is*

$$\tilde{\mathbf{g}}^k = \tilde{\mathbf{g}}^{k-1} + \omega D^{-1}[\mathbf{g} - A\tilde{\mathbf{g}}^{k-1}]. \quad (5.6)$$

- If one considers a sequence of relaxation parameters $\{\omega_k\}_k$, then the damped block Jacobi method is equivalent to Richardson's method [47], namely $\tilde{\mathbf{g}}^k = \tilde{\mathbf{g}}^{k-1} + \omega_k [\mathbf{g} - A\tilde{\mathbf{g}}^{k-1}]$.
- Assume that $v^0 \in H$ and $\tilde{g}_j^0 = \tau_j v^0$ and $\omega = \frac{1}{J}$. Then the APMR (5.4) is equivalent to (5.6).

Proof. The first statement follows by a direct calculation on (5.6). The second statement follows easily by recalling that $D = I$ and considering the parameter ω depending on the iteration k . Setting $\omega = \frac{1}{J}$ and recalling the operators D , L , and U given in Section 3.3, we can write (5.5) in the pointwise form $\tilde{g}_j^k = \left(1 - \frac{1}{J}\right)\tilde{g}_j^{k-1} + \frac{1}{J}\left[g_j - \sum_{\ell=1, \ell \neq j}^J \tau_\ell G_\ell(\tilde{g}_\ell^{k-1})\right]$. Then similar arguments as in Theorem 3.4 imply that this is equivalent to the APMR in (5.4). ■

We have seen that the APMR can be regarded as a relaxed (or damped) block Jacobi method or as Richardson's method with the specific choice $\omega_k = \omega = \frac{1}{J}$. This choice, motivated in [38] by a Hilbert projection analysis, is not guaranteed to be optimal. In what follows we derive under suitable conditions, an explicit formula for the optimal parameter ω^* in discrete settings. Denoting by $G_J(\omega)$ the damped-block Jacobi iteration matrix obtained, for example, by a boundary-element discretization of (3.13) and (5.5), the optimal parameter ω^* is then given by $\omega^* = \arg \min_{\omega \in [0,1]} \rho(G_J(\omega))$, where $\rho(G_J(\omega))$ is the spectral radius of $G_J(\omega)$. We have the following results.

Theorem 5.3. *Let $\lambda_k(G_J(1))$, for $k = 1, 2, \dots$, be the eigenvalues of the block Jacobi iteration matrix. Assume that $\lambda_k(G_J(1)) \in \mathbb{R}$ for any k and that $\rho(G_J(1)) < 1$, that is the block Jacobi method converges. Then $\omega^* = \frac{2}{2 - (\lambda_{\max} + \lambda_{\min})}$, where λ_{\max} and λ_{\min} are the maximum and minimum eigenvalues of $G_J(1)$.*

Proof. Recalling (5.5), we observe that the damped block Jacobi iteration matrix has the form $G_J(\omega) = (1 - \omega)I + \omega G_J(1)$. Hence the spectral radius $\rho(G_J(\omega))$ has the form $\rho(G_J(\omega)) = \max\{|1 - \omega(1 - \lambda_{\max})|, |1 - \omega(1 - \lambda_{\min})|\}$, and the optimal parameter ω^* is the point where the two straight functions $|1 - \omega(1 - \lambda_{\max})|$ and $|1 - \omega(1 - \lambda_{\min})|$ intersect, that is $\omega = \frac{2}{2 - (\lambda_{\max} + \lambda_{\min})}$. ■

Theorem 5.3 provides the optimal parameter under the assumption that the block Jacobi method converges. If block Jacobi does however not converge, can ω be used to make it convergent? In other words, if an eigenvalue of $G_J(1)$ is such that $|\lambda_k(G_J(1))| \geq 1$, is there a choice of ω capable to correct this behavior? To study this unfavorable case, we have to distinguish different cases:

- if $\lambda_k(G_J(1)) = 1$, then $\lambda_k(\omega) = 1 - \omega(1 - \lambda_k(G_J(1))) = 1$ for any ω . Hence, there exists no $\omega \in \mathbb{R}$ such that $|\lambda_k(\omega)| < 1$.
- if $\lambda_k(G_J(1)) = -1$, then $\lambda_k(\omega) = 1 - 2\omega$, which implies that $|\lambda_k(\omega)| < 1$ if and only if $0 < \omega < 1$.
- if $\lambda_k(G_J(1)) < -1$, then $|\lambda_k(\omega)| = |1 - \omega(1 - \lambda_k(G_J(1)))| < 1 \Leftrightarrow 0 < \omega < \frac{2}{1 - \lambda_k(G_J(1))}$.
- if $\lambda_k(G_J(1)) > 1$, then $|\lambda_k(\omega)| = |1 - \omega(1 - \lambda_k(G_J(1)))| < 1 \Leftrightarrow \frac{2}{1 - \lambda_k(G_J(1))} < \omega < 0$.

We can summarize these facts in the following result.

Theorem 5.4. *Let $G_J(1)$ and $G_J(\omega)$ be the block Jacobi and damped block Jacobi iteration matrices and assume that $|\lambda_k(G_J(1))| \geq 1$ for a given k . We have that*

- if $\lambda_k(G_J(1)) = -1$, then $|\lambda_k(\omega)| < 1$ for any $\omega \in (0, 1)$.
- if $\lambda_k(G_J(1)) = 1$, then $|\lambda_k(\omega)| = 1$ for any $\omega \in \mathbb{R}$.
- if $\lambda_k(G_J(1)) < 0$, then $|\lambda_k(\omega)| < 1$ if and only if $0 < \omega < \frac{2}{1 - \lambda_k(G_J(1))}$.

- if $\lambda_k(G_J(1)) > 1$, then $|\lambda_k(\omega)| < 1$ if and only if $\frac{2}{1-\lambda_k(G_J(1))} < \omega < 0$.

Theorem 5.4 shows that if block Jacobi does not converge, it is not always possible to find a parameter $\omega \in (0, 1]$ that makes the method convergent. If the eigenvalues $\lambda_k(G_J(1))$ are such that $\lambda_k(G_J(1)) < 1$, i.e. there can be arbitrarily large negative eigenvalues, a case that we have observed numerically, then a similar proof as for Theorem 5.3 allows us to obtain the following result.

Theorem 5.5. *Let $G_J(1)$ and $G_J(\omega)$ be the block Jacobi and damped-block Jacobi iteration matrices. Assume that the eigenvalues $\lambda_k(G_J(1))$ are real and such that $\lambda_k(G_J(1)) < 1$. The optimal parameter is $\omega^* = \frac{2}{2-(\lambda_{\max}+\lambda_{\min})}$, where λ_{\max} and λ_{\min} are the maximum and minimum eigenvalues of $G_J(1)$.*

5.2. Successive-Over-Relaxation method (SOR)

We now rewrite the AltMR as an SOR method, which converges in general much faster for a well chosen relaxation parameter [54]. SOR for the system (3.13) is

$$\tilde{\mathbf{g}}^k = (D + \omega L)^{-1} [\omega \mathbf{g} + (-\omega U + (1 - \omega)D)\tilde{\mathbf{g}}^{k-1}]. \quad (5.7)$$

Recalling the form of D , L , and U given in Section 3.3, (5.7) written component-wise is

$$\tilde{g}_j^k = (1 - \omega)\tilde{g}_j^{k-1} + \omega \left[g_j - \sum_{\ell=1}^{j-1} \tau_j G_\ell(\tilde{g}_\ell^k) - \sum_{\ell=j+1}^J \tau_j G_\ell(\tilde{g}_\ell^{k-1}) \right]. \quad (5.8)$$

Defining $v_j^k := G_j(\tilde{g}_j^k)$, this iteration can be rewritten in the volume form,

$$\Delta v_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad v_j^k = 0 \text{ on } \partial \Omega, \quad v_j^k = (1 - \omega)v_j^{k-1} + \omega \left[g_j - \sum_{\ell=1}^{j-1} v_\ell^k - \sum_{\ell=j+1}^J v_\ell^{k-1} \right] \text{ on } \partial O_j, \quad (5.9)$$

with $u^k = \sum_{j=1}^J v_j^k$. This is a new formulation of the method of reflections that can be also written in terms of the (more usual) d_j^k , that is

$$\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^k = 0 \text{ on } \partial \Omega, \quad d_j^k = (1 - \omega)d_j^{k-1} - \omega \left[\sum_{\ell=1}^{j-1} d_\ell^k + \sum_{\ell=j+1}^J d_\ell^{k-1} \right] \text{ on } \partial O_j, \quad (5.10)$$

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j, \quad d_j^1 = 0 \text{ on } \partial \Omega, \quad d_j^1 = g_j - u^0 - \omega \sum_{\ell=1}^{j-1} d_\ell^1 \text{ on } \partial O_j, \quad (5.11)$$

and $u^k = u^{k-1} + \omega \sum_{j=1}^J d_j^k = u^0 + \omega \sum_{n=1}^k \sum_{j=1}^J d_j^n$. Notice that (5.10)-(5.11) and (5.8) are equivalent, because (5.8) is the equivalent substructured form of (5.9) and the equivalence between (5.9) and (5.10)-(5.11) is proved in the following theorem.

Theorem 5.6. *Consider the sequences $\{v_j^k\}_{k \in \mathbb{N}} \subset H_j$ and $\{d_j^k\}_{k \in \mathbb{N}^+} \subset H_j$ for $j = 1, \dots, J$, and $\{u^k\}_{k \in \mathbb{N}} \subset H$ with $u^0 = \sum_{j=1}^J v_j^0$. Assume that $v_j^k = v_j^{k-1} + \omega d_j^k$ for $k \geq 1$ and $j = 1, \dots, J$ and $\omega > 0$. Then the following statements are equivalent:*

- the d_j^k , $j = 1, \dots, J$, solve (5.10)-(5.11) and $u^k = u^{k-1} + \omega \sum_{j=1}^J d_j^k = u^0 + \omega \sum_{n=1}^k \sum_{j=1}^J d_j^n$.
- the v_j^k , $j = 1, \dots, J$, solve (5.9) and $u^k = \sum_{j=1}^J v_j^k$.

Proof. (a) \Rightarrow (b): first, using $v_j^k = v_j^{k-1} + \omega d_j^k$, we write that

$$u^k = u^{k-1} + \omega \sum_{j=1}^J d_j^k = u^0 + \sum_{n=1}^k \sum_{j=1}^J (v_j^n - v_j^{n-1}) = \sum_{j=1}^J v_j^k. \quad (5.12)$$

Notice that $v_j^k \in H_j$, hence it is sufficient to prove that the boundary condition on ∂O_j in (5.9) holds. Using the transmission condition of (5.10) and $u^{k-1} = u^{k-2} + \omega \sum_{\ell=1}^J d_\ell^{k-1}$, we obtain

$$d_j^k = d_j^{k-1} - \omega \sum_{\ell=1}^{j-1} d_\ell^k - \omega \sum_{\ell=1}^J d_\ell^{k-1} + \omega \sum_{\ell=1}^{j-1} d_\ell^{k-1} = d_j^{k-1} - \omega \sum_{\ell=1}^{j-1} d_\ell^k - (u^{k-1} - u^{k-2}) + \omega \sum_{\ell=1}^{j-1} d_\ell^{k-1},$$

which implies that $d_j^k + u^{k-1} + \omega \sum_{\ell=1}^{j-1} d_\ell^k = \dots = d_j^1 + u^0 + \omega \sum_{\ell=1}^{j-1} d_\ell^1 = g_j$, where we used the transmission condition of (5.11). Hence, we have obtained that $d_j^k = g_j - u^{k-1} - \omega \sum_{\ell=1}^{j-1} d_\ell^k$. Using this equation, $v_j^k = v_j^{k-1} + \omega d_j^k$ and (5.12), we obtain

$$\frac{1}{\omega}(v_j^k - v_j^{k-1}) = d_j^k = g_j - u^{k-1} - \omega \sum_{\ell=1}^{j-1} d_\ell^k = g_j - \sum_{\ell=1}^{j-1} v_\ell^k - \sum_{\ell=j}^J v_\ell^{k-1},$$

which implies that $v_j^k = v_j^{k-1} + \omega g_j - \omega \sum_{\ell=1}^{j-1} v_\ell^k - \omega \sum_{\ell=j}^J v_\ell^{k-1}$, and our claim follows.

(b) \Rightarrow (a): using $d_j^k = \frac{1}{\omega}(v_j^k - v_j^{k-1})$, we obtain

$$u^k = \sum_{j=1}^J v_j^k = u^0 + \sum_{j=1}^J (v_j^k - v_j^0) = u^0 + \sum_{n=1}^k \sum_{j=1}^J (v_j^n - v_j^{n-1}) = u^0 + \omega \sum_{n=1}^k \sum_{j=1}^J d_j^n,$$

which implies that $u^k = u^{k-1} + \omega \sum_{j=1}^J d_j^k$. Now, since $d_j^k = \frac{1}{\omega}(v_j^k - v_j^{k-1})$ we have that $d_j^k \in H_j$. It remains to show that the transmission conditions of (5.10)-(5.11) hold. For $k = 1$ we have

$$d_j^1 = \frac{1}{\omega}(v_j^1 - v_j^0) = g_j - \sum_{\ell=1}^{j-1} v_\ell^1 - \sum_{\ell=j}^J v_\ell^0 = g_j - \omega \sum_{\ell=1}^{j-1} d_\ell^1 - \sum_{\ell=1}^{j-1} v_\ell^0 - \sum_{\ell=j}^J v_\ell^0 = g_j - \omega \sum_{\ell=1}^{j-1} d_\ell^1 - u^0.$$

In the case $k > 1$, we write the transmission condition of (5.9) on ∂O_j for k and $k-1$, and subtracting them term by term, dividing by ω , and recalling that $d_j^k = \frac{1}{\omega}(v_j^k - v_j^{k-1})$ we obtain the transmission condition of (5.10). \blacksquare

A good choice of the parameter ω is not trivial. In general one would consider the optimal parameter ω^* , that is the parameter that minimizes the convergence factor of the SOR method. From a discrete point of view, namely when one is interested in solving a linear system $A\tilde{\mathbf{g}} = \mathbf{g}$ using a stationary iterative method of the form $\tilde{\mathbf{g}}^{k+1} = M^{-1}N\tilde{\mathbf{g}}^k + M^{-1}\mathbf{g}$, there are several results in the literature for point relaxation (in contrast to block relaxation here). A famous result proved by Kahan [33] says that for the SOR method one has $\rho(M^{-1}N) \geq |1 - \omega|$, which implies that a necessary condition for convergence is $\omega \in (0, 2)$. In the case the matrix A has the so called Property A, David Young proved in [54] that the optimal parameter, that is the parameter that minimizes the spectral radius $\rho(G_{\text{SOR}}(\omega))$ with respect to ω , where $G_{\text{SOR}}(\omega)$ is the SOR-iteration matrix, is given by $\omega^* = \frac{2}{1 + \sqrt{1 - \rho(G_J)^2}}$, where G_J is the iteration matrix of the corresponding point-Jacobi method applied to the same linear system. A discrete form of our problem (3.13) for $J = 2$ is characterized by a matrix A of the form $A = \begin{bmatrix} I_1 & G_2 \\ G_1 & I_2 \end{bmatrix}$, where I_1 and I_2 are identity matrices. This shows that A is exactly in the form required to have the Property A. Therefore, we can apply the theory developed by David Young to get an explicit formula for the optimal parameter. This observation is not in general true for $J > 2$. Moreover, to compute ω^* one would need the spectral radius of the block Jacobi method (the PMR). This is not an easy task, even for $J = 2$. However, we can estimate $\rho(G_J)$ at a continuous level using maximum principle arguments, as we show in Section 3.5. Numerical experiments in Section 7 show that this leads to a very good estimate of ω^* for $J = 2$. In the case that $J > 2$ the optimal parameter depends strongly on the geometry and we observed that when the distance between the objects is sufficiently large the optimal choice is $\omega^* \approx 1$.

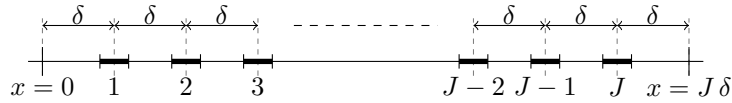


FIGURE 6.1. Geometry of the one-dimensional example used to study scalability of the AltMR. In this figure, δ is the distance between the centers of the J equidistant objects that are represented by the thick segments.

6. Scalability analysis and coarse correction

An iterative method is said to be optimal, if its rate of convergence is independent of the size of the system; [51, Definition 1.2 (p. 9)]. A domain decomposition iterative method is said to be scalable, if its rate of convergence does not deteriorate when the number of subdomains grows; [51, Definition 1.3 (p. 17)]. The number of subdomains can grow in two ways: either the domain Ω is fixed and the subdomains shrink, but by mesh refinement the work per subdomain remains the same¹, or the subdomain and mesh size is kept fixed, and Ω grows. In both cases the work per iteration is constant, since the subdomain solves are performed in parallel. The first case of scalability (Ω fixed) is widely studied in the literature, see, e.g., [51] and references therein. For the second case (Ω growing) only recently an interesting property was observed for the classical one level Schwarz method in [6], and then theoretically investigated in [8, 9, 10], for classical one-level Neumann-Neumann, Dirichlet-Neumann and optimized-Schwarz methods see [7]. If a domain decomposition method is not scalable, then a coarse correction can make it scalable.

In this section, we study optimality and scalability properties for the methods of reflections. We will see that methods of reflections are not scalable, but they are optimal. The optimality is demonstrated by direct numerical experiments (see Table 7.3), while the non-scalability is shown by a simple one-dimensional example (see Figure 6.1). Notice that, if the domain Ω is fixed and one increases the number of objects, then the distance between them necessarily reduces, and we have seen in Section 7.3 for $J = 2$ that the contraction factor deteriorates in this case. The same behavior has been observed, e.g., in [38, 11], and thus a coarse-correction is needed in that case to restore scalability. But even if the distance between the objects is kept constant, by increasing the number of fixed-sized holes (hence increasing Ω) the methods of reflections are not scalable. To see this, we consider a one-dimensional problem characterized by J equidistant objects (subintervals); see Figure 6.1. This is a finite-dimensional problem because the unknowns are the Dirichlet data $\mathbf{g} \in \mathbb{R}^{2J}$ on the $2J$ extrema of the objects. The problem can then be written in a form $A\mathbf{g} = \mathbf{g}$, where $\mathbf{g} \in \mathbb{R}^{2J}$ and $A \in \mathbb{R}^{2J \times 2J}$. Using the equivalence result proved in Theorem 3.4, one can easily construct the block Gauss-Seidel (AltMR) iteration matrix, that is $G_{\text{GS}} = -(D+L)^{-1}U$ and numerically compute its spectral radius for increasing number of objects J . The result is given in Figure 6.2, which clearly shows that the AltMR does not scale because $\rho(G_{\text{GS}})$ deteriorates for growing J . The heuristic reason for this behavior is that, once the approximate solution is corrected on one object, say the j th one, then the procedure continues to correct it sequentially on all the other objects. Every correction induces an additional error in the approximate solution on the j th object. Therefore, when J increases, more and more objects have to be corrected, which adds more and more error induced by the alternating correction procedure on the j th object. Similar arguments apply also for the other methods of reflections. We thus need a coarse correction to obtain a scalable method.

¹If one does not refine the mesh, the work per subdomain diminishes, and we get to what is called strong scalability. Strong scalability is not asymptotic, because eventually no work is left per subdomain. Strong scalability does not apply to the methods of reflections, since one does not lower the resolution on the objects when more and more objects are simulated, so we only consider here the scalability explained in the text, which is often called weak scalability.

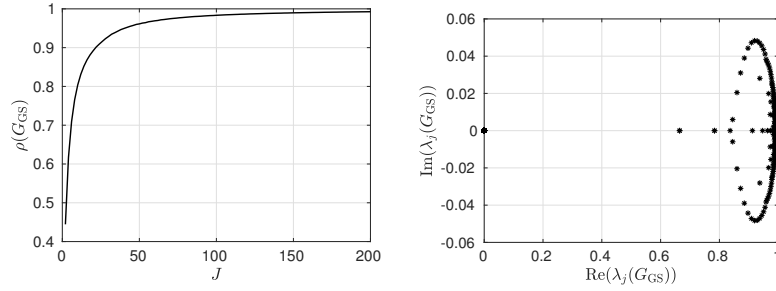


FIGURE 6.2. Left: Spectral radius of the AltMR as a function of the number of objects. Right: Example of the spectrum of the (one-dimensional) operator $G_{\text{GS}} = -(D + L)^{-1}U$. Notice that many eigenvalues are clustered around 1.

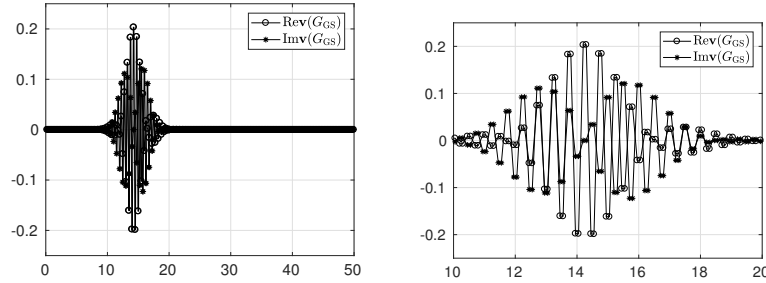


FIGURE 6.3. Example of an eigenvector of the one-dimensional iteration matrix G_{GS} corresponding to an eigenvalue $\lambda \approx 1$. The right plot shows a zoom into the highly oscillating area of the eigenvector depicted in the left plot.

A spectral analysis of the one-dimensional iteration matrix G_{GS} reveals that many of its eigenvalues are clustered around 1, see Figure 6.2 (right), and that the corresponding eigenvectors have a special structure: they globally oscillate with (local) maxima and minima attained on the objects, and many of them have constant value on the objects; see Figure 6.3. This suggests that a coarse-correction has mainly to deal with errors on the boundaries of the objects. Therefore, to design our coarse-space we denote by $\psi_{j,n}$ for $n = 0, 1, \dots$ the eigenfunction of the Laplace-Beltrami operator on the j th object ∂O_j , for $j = 1, \dots, J$, and introduce functions $\varphi_{j,n}$ as solution to the Dirichlet problem

$$\Delta \varphi_{j,n} = 0 \text{ in } \Omega \setminus \cup_{\ell=1}^J \partial O_\ell, \quad \varphi_{j,n} = \psi_{j,n} \text{ on } \partial O_j, \quad \varphi_{j,n} = 0 \text{ on } \partial \Omega \cup \left(\cup_{\ell=1, \ell \neq j}^J \partial O_\ell \right), \quad (6.1)$$

for $j = 1, \dots, J$ and $n = 0, 1, \dots$. Notice that $\varphi_{j,n} \in H$ for any j and n , and the functions $\psi_{j,n}$ are classical Fourier-basis functions in two dimensions and spherical harmonics in three dimensions. Our coarse-space is then defined as

$$V_c^N := \text{span}_{j=1, \dots, J, n=0, 1, \dots, N} \{ \varphi_{j,n} \},$$

where N is the number of eigenfunctions considered. Notice that the dimension of V_c^N is proportional to N . It is clear that the coarse space V_c^N has mainly information condensed on the boundaries of the objects, similar to the Spectral Harmonically Enriched Multiscale coarse space SHEM in domain decomposition [21, 20], which contains mainly information on the interfaces between subdomains, see also [19, 18]. It is important to remark that the construction of each function $\varphi_{j,n}$ would require the solution of problem (6.1), which requires the same computational effort of the original problem (2.3). However, this is not needed because the substructured formulation introduced in Section 3.3 allows

us to work directly on the boundary of the objects: our coarse space contains thus spectral approximations of the substructured problem. For this reason, we introduce the restriction operator and its corresponding adjoint (prolongation or extension) operator as

$$R := \begin{bmatrix} \mathcal{F}_1^N & & & \\ & \mathcal{F}_2^N & & \\ & & \ddots & \\ & & & \mathcal{F}_J^N \end{bmatrix} \quad \text{and} \quad R^* = \begin{bmatrix} (\mathcal{F}_1^N)^* & & & \\ & (\mathcal{F}_2^N)^* & & \\ & & \ddots & \\ & & & (\mathcal{F}_J^N)^* \end{bmatrix},$$

where $\mathcal{F}_j^N : L^2(\partial O_j) \rightarrow \mathbb{R}^N$ is given by

$$\mathcal{F}_j^N(w) = [c_{1,j} \quad \dots \quad c_{N,j}]^\top =: \mathbf{c}_j,$$

with $c_{n,j} = \int_{\partial O_j} w \psi_{j,n}$ for any $w \in L^2(\partial O_j)$ and $j = 1, \dots, J$, and $(\mathcal{F}_j^N)^* : \mathbb{R}^N \rightarrow L^2(\partial O_j)$ is

$$(\mathcal{F}_j^N)^*(\mathbf{c}_j) = \sum_{n=1}^N c_{n,j} \psi_{j,n},$$

for $\mathbf{c}_j \in \mathbb{R}^N$. These operators allow us to restrict the operator A introduced in (3.13) and (3.14) on the coarse space V_c^N as $A_c = RAR^* \in \mathbb{R}^{JN \times JN}$.

We are now ready to state our two-level method of reflection: given an approximation $\tilde{\mathbf{g}}^k$ to $\tilde{\mathbf{g}}$ at the iteration k , a coarse-corrected method of reflections step is defined as

$$\begin{aligned} \tilde{\mathbf{g}}^{k+1/2} &:= G_{\text{MR}} \tilde{\mathbf{g}}^k + M_{\text{MR}}^{-1} \mathbf{g} && \text{(one-level MR step),} \\ \tilde{\mathbf{r}}^{k+1/2} &:= \mathbf{g} - A \tilde{\mathbf{g}}^{k+1/2} && \text{(compute the residual),} \\ \tilde{\mathbf{g}}_c^{k+1/2} &:= A_c^{-1} R \tilde{\mathbf{r}}^{k+1/2} && \text{(compute the correction),} \\ \tilde{\mathbf{g}}^{k+1} &:= \tilde{\mathbf{g}}^{k+1/2} + R^* \tilde{\mathbf{g}}_c^{k+1/2} && \text{(correct } \tilde{\mathbf{g}}^{k+1/2}\text{),} \end{aligned}$$

where G_{MR} is the one-level method of reflection operator and M_{MR} the corresponding preconditioning matrix, which can be, e.g., G_{GS} (with $M_{\text{MR}} = (D + L)$) for the AltMR or G_{J} (with $M_{\text{MR}} = D$) for the PMR. A direct calculation reveals that the two-level method of reflections iteration operator is

$$G_{\text{MR-c.c.}} = [I - R^* A_c^{-1} R A] G_{\text{MR}}.$$

A coarse corrected method of reflection is given in Algorithm 5.

Algorithm 5 Method of Reflections with Coarse Correction

Require: K (maximum number of iterations), tol (tolerance), $\tilde{\mathbf{g}}^0$ (initial guess).

Require: $f_{\text{MR}}(\mathbf{v}, \mathbf{w}) := G_{\text{MR}} \mathbf{v} + M_{\text{MR}}^{-1} \mathbf{w}$ (function that performs one step of a method of reflection MR),

Require: \mathbf{g} (data of the problem).

- 1: Compute $\tilde{\mathbf{r}}^0 = \mathbf{g} - A \tilde{\mathbf{g}}^0$ and set $k = 0$.
 - 2: **while** $k < K$ and $\|\tilde{\mathbf{r}}^k\| > tol$ **do**
 - 3: Set $k = k + 1$.
 - 4: $\tilde{\mathbf{g}}^k = f_{\text{MR}}(\tilde{\mathbf{g}}^{k-1}, \mathbf{g})$.
 - 5: $\tilde{\mathbf{r}}^k = \mathbf{g} - A \tilde{\mathbf{g}}^k$.
 - 6: $\tilde{\mathbf{g}}_c = A_c^{-1} R \tilde{\mathbf{r}}^k$.
 - 7: $\tilde{\mathbf{g}}^{k+1} = \tilde{\mathbf{g}}^k + R^* \tilde{\mathbf{g}}_c$.
 - 8: **end while**
-

Notice that like in domain decomposition algorithms, each iteration is in general performed in parallel, and thus the corresponding iteration cost is the order of one object solve. This would in principle imply scalability in terms of computation time. However, like in domain decomposition

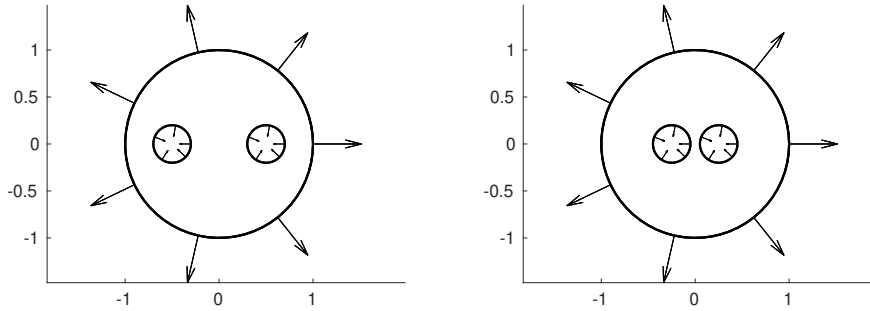


FIGURE 7.1. Geometry of the problem, domain with some normal unit vectors. Left: objects relatively far from each other, right: objects relatively close to each other.

algorithms, Algorithm 5 requires the solution of the coarse problem. If the dimension of the coarse space is proportional to J , then this solution requires in principle $O(J^3)$ flops. Therefore, for large J the cost of the coarse correction can start dominating the cost of solving one object, and then, as in domain decomposition, our two-level framework would need to be extended to a multi-level setting to remain scalable, which is however beyond the scope of this paper.

7. Numerical experiments

We now present some numerical tests to illustrate the results we obtained, and to explore cases not covered by the theory we developed. We start with the case of two objects, and then investigate the case of three objects. We then study the scalability of the method with respect to the number of objects and finally consider the method as a preconditioner for the GMRES algorithm. We used a publicly available package² of Matlab functions to solve the integral equations of the problem at collocation points by the Nyström method.

7.1. Two objects

We first consider the case of two objects of radius $r = 0.2$, in the unit disc, for which we have theoretical results. The theoretical value of the optimal ω^* for the SOR variant of the method of reflections is given by

$$\omega^*(r, \delta) = \frac{2}{1 + \sqrt{1 - \rho_{\text{PMR}}(r, \delta)^2}}. \quad (7.1)$$

With this choice of ω , the theoretical convergence factor is

$$\rho_{\text{SOR}}(r, \delta)_{\omega=\omega^*} = \omega^*(r, \delta) - 1 \quad (7.2)$$

In a first experiment, we consider two cases where the distance between the objects is either relatively large, that is $\delta = 0.5$, or relatively small that is $\delta = 0.25$, see Figure 7.1. We consider both SOR and damped PMR. In the case of SOR, ω^* is given by Equation (7.1), where, in our 2 object case, $\rho_{\text{PMR}}(r, \delta)$ can be computed numerically and estimated theoretically. Alternatively the latter quantity can be estimated using formulas provided in Theorem 4.4 and Corollary 3.7. Results are presented in Figure 7.2. We see that the agreement between the theoretical and numerical values of ω^* is good when the objects are far from each other, and less accurate when the objects are close to each other. We also see that the relaxation parameter ω^* is only close to 1 in the first case, which means that the SOR variant provides a real improvement with respect to the standard AltMR when the objects are

²Integral Equation Solver (<http://www.mathworks.com/matlabcentral/fileexchange/34241>) by Alexandre Munier and Bruno Pinçon, MATLAB Central File Exchange. Retrieved February 15, 2016.

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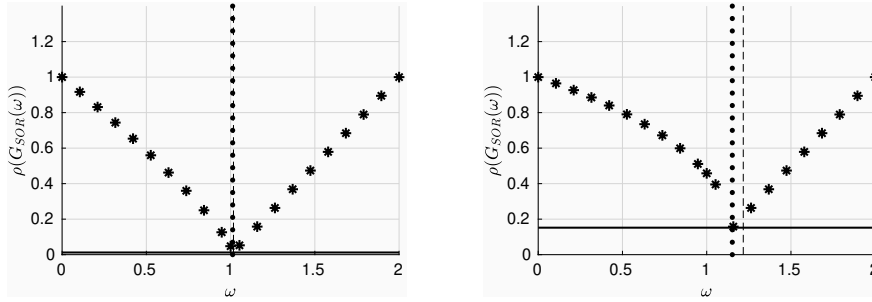


FIGURE 7.2. Contraction factor of the SOR variant of the method of reflections with respect to ω (stars), theoretical values of ω^* given by (7.1) where $\rho_{\text{PMR}}(r, \delta)$ is obtained either using Theorem 4.4 and Corollary 3.7 (dashed line), or computed numerically (dotted line). In the latter case, the optimal contraction factor is computed by Equation (7.2) (solid line). Left: the objects are far away from each other. Right: the objects are close to each other.

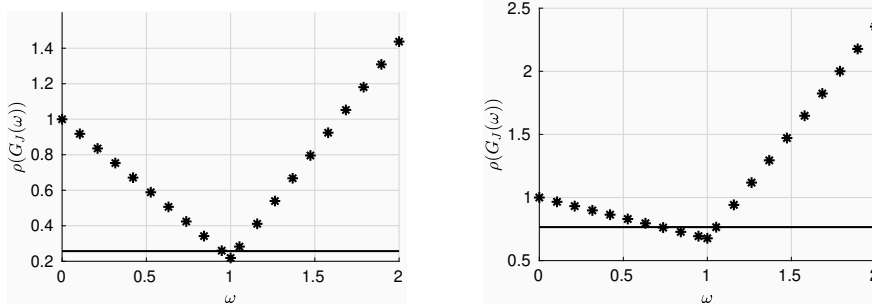


FIGURE 7.3. Contraction factor of the damped PMR with respect to ω (stars), theoretical estimate of the contraction factor (solid line) given by Theorem 4.4 and Corollary 3.7. Left: the objects are far from each other. Right: the objects are close to each other.

relatively close to each other. In the case of PMR, we observe that the damping does not improve the contraction factor, see Figure 7.3.

The theoretical estimate of the contraction factor is rather accurate and does not seem to depend on the distance between the objects. Note also that the eigenvalues of the PMR iteration matrix form a symmetric set with respect to zero. Indeed, one has $G_J = -D^{-1}(L + U) = -\begin{bmatrix} 0 & \tilde{G} \\ \tilde{G} & 0 \end{bmatrix}$, so that

$$\det(G_J - \lambda I) = \det \begin{bmatrix} -\lambda I & -\tilde{G} \\ -\tilde{G} & -\lambda I \end{bmatrix} = \det(-\lambda I) \det \left(-\lambda I + \frac{1}{\lambda} \tilde{G}^2 \right) = \det \left(-\lambda^2 I + \tilde{G}^2 \right).$$

As a consequence, if λ is an eigenvalue, then also $-\lambda$ is an eigenvalue. Because of Theorem 5.3, we find that in this case $\omega^* = 1$, as observed in Figure 7.3.

Finally, we compare theoretical and numerical values of the contraction factors both for AltMR and PMR when the distance δ between the objects varies; the results are shown in Figure 7.4, and we see a good agreement between theory and numerics.

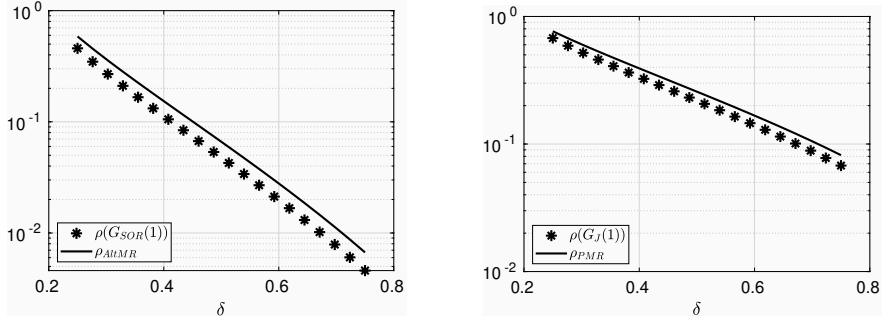


FIGURE 7.4. Contraction factors of AltMR (left) and PMR (right) with respect to δ and theoretical contraction factors (solid lines) given by Theorems 3.6 and 4.4 and Corollary 3.7, compared with numerical contraction factors (stars).

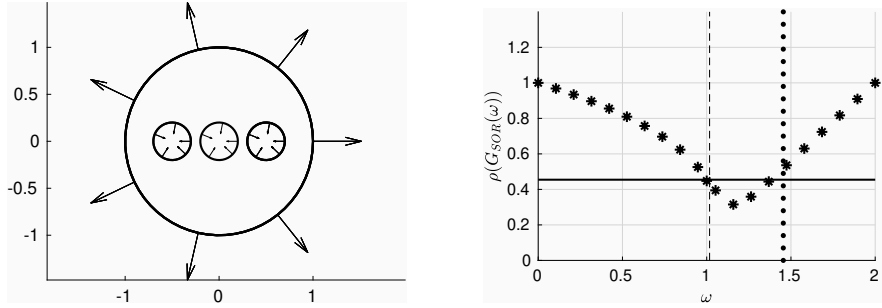


FIGURE 7.5. Left: geometry of the problem, domain with some normal unit vectors. Right: Contraction factor of the SOR variant with respect to ω (stars), theoretical values of ω^* given by (7.1) where $\rho_{\text{PMR}}(r, \delta)$ is obtained either from Theorem 4.4 and Corollary 3.7 (dashed line) or computed numerically (dot line). In the latter case, the optimal contraction factor is computed by Equation (7.2) (solid line).

7.2. Three objects

We now consider the case of three objects of radius $r = 0.2$ in the unit disc, for which we do not have theoretical results. For the sake of simplicity, we restrict ourselves to SOR, and repeat the experiments done to obtain Figure 7.2. The results are presented in Figure 7.5. We observe again that SOR provides a real improvement compared to the standard AltMR, in the sense that the observed optimal ω differs from 1. We however also see that the estimates obtained with Equation (7.1) are not accurate now.

7.3. Scalability with respect to the number of objects

In this test, we study the scalability of the AltMR and PMR when the number of objects varies, and the effect of the coarse correction on both methods. We consider for a given $L \in \mathbb{N}^+$ a square of size $L \times L$ with rounded corners containing $J = L^2$ objects of radius $r = 0.1$, see Figure 7.6. We then evaluate numerically the contraction factor for both methods, with and without coarse correction, see Figure 7.7. We consider here the coarse space $V_c^0 = \text{span} \{\varphi_{j,0}\}_{j=1,\dots,J}$ of dimension J , where we use only the first (constant) Fourier mode $\psi_{j,0}$ for each object, see (6.1). We observe that the coarse correction significantly reduces the contraction factor, by approximately one order of magnitude for the AltMR and even more for the PMR, where the coarse correction leads to a convergence factor below 1, so that the PMR with coarse correction remains convergent even in cases where the standard PMR diverges.

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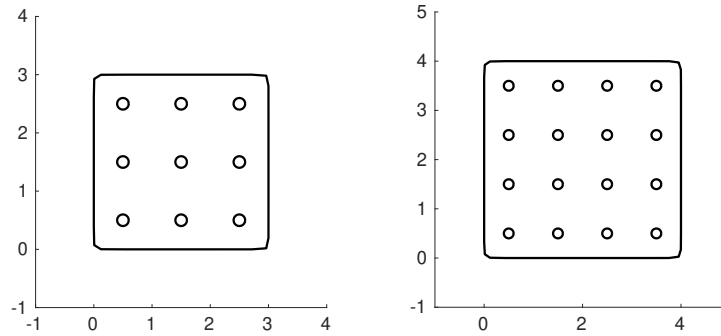


FIGURE 7.6. Geometry of the problem for $L = 3$ (left) and $L = 4$ (right).

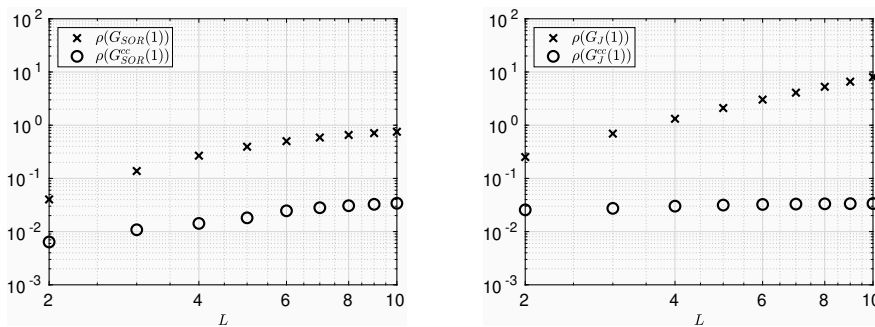


FIGURE 7.7. Contraction factors of the AltMR (left) and PMR (right) with respect to the number of objects. Contraction factor without (crosses) and with (circles) coarse correction.

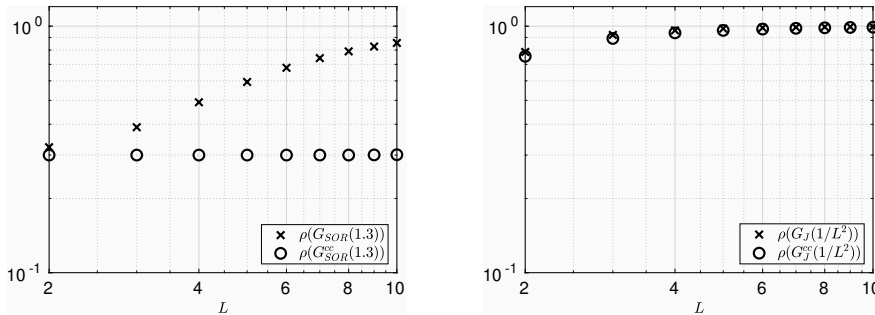


FIGURE 7.8. Contraction factors of the SOR variant with $\omega = 1.3$ (left) and APMR (right) with respect to the number of objects. Contraction factor without (crosses) and with (circles) coarse correction.

We next repeat these two tests with the method of reflection variants with relaxation, namely, with the SOR variant and the APMR, see Figure 7.8. We see that the coarse correction significantly improves the performance of the SOR variant, but not the APMR. It even seems that the convergence rate remains unchanged in the case of APMR. This is due to the relaxation parameter $\omega = \frac{1}{J}$ that decays as J grows.

Next, we solve the problem using GMRES and employing the methods of reflections as preconditioners, both with and without coarse correction. For reference, we compare the performance of these methods with the coarse-corrected methods of reflections used as stationary iterative methods; the

# Objects ($r = 0.1$)	4	9	16	25	36	49	64	81	100	121	144
GMRES-PMR	6	9	11	14	16	17	19	20	21	23	24
GMRES-AltMR	5	8	11	14	16	19	22	24	27	30	32
GMRES-PMR c.c.	3	3	3	3	3	3	3	3	3	3	3
GMRES-AltMR c.c.	3	3	3	3	3	4	4	4	4	4	4
PMR c.c.	8	8	9	9	9	9	9	9	9	9	9
AltMR c.c.	6	7	8	8	9	9	9	9	10	9	9

TABLE 7.1. Number of GMRES iterations necessary to get a residual smaller than 10^{-10} . The different columns correspond to different numbers of objects and show the number of iterations performed by GMRES and various preconditioners: AltMR, PMR with coarse correction, and AltMR with coarse correction. In the two last lines, we show the number of iterations required by coarse corrected AltMR and PMR used as stationary iterations.

# Objects ($r = 0.3$)	4	9	16	25	36	49	64	81	100	121	144
GMRES-PMR	10	15	19	22	25	28	30	32	35	37	38
GMRES-AltMR	7	11	15	19	23	28	32	36	40	44	48
GMRES-PMR c.c.	5	6	6	6	6	6	7	6	7	7	7
GMRES-AltMR c.c.	5	6	6	7	7	8	8	8	8	8	8
PMR c.c.	18	16	21	18	22	20	22	22	22	23	23
AltMR c.c.	12	15	18	20	22	23	23	23	23	25	26

TABLE 7.2. Same experiment setting as in Table 7.1, but for a larger object radius $r = 0.3$.

mesh size Δx	0.3142	0.1571	0.0785	0.0393	0.0196	0.0098	0.0049	0.0025	0.0012
GMRES-PMR	9	9	9	9	8	9	9	8	8
GMRES-AltMR	8	8	8	8	8	8	8	8	8
GMRES-PMR c.c.	3	3	3	3	3	3	3	3	3
GMRES-AltMR c.c.	3	3	3	3	3	3	3	3	3
PMR c.c.	8	8	8	8	8	8	8	8	8
AltMR c.c.	7	7	7	7	7	7	7	7	8

TABLE 7.3. Same experiment setting as in Table 7.1, but for a fixed number of objects $J = L^2 = 9$ and different values of mesh size Δx . In particular, Δx denotes the mesh size used to discretize uniformly both the external boundary $\partial\Omega$ and the objects ∂O_j for $j = 1, \dots, J$.

required iteration numbers are summarized in Table 7.1. We observe that with GMRES now the PMR preconditioner leads to a convergent method, even though as a stationary iterative method it was not convergent. This is very similar to the additive Schwarz method which is also not converging as a stationary iterative method [15, 17]. We also see that with our coarse correction one obtains scalability with respect to the number of objects. This scalability also holds for the stationary iterative variants. We next repeat the experiment using the same geometry but increasing the radius of the objects to $r = 0.3$, which implies that the distance between the objects is smaller, and thus a slower convergence rate of the methods of reflections. The number of iterations are summarized in Table 7.2. We see that indeed iteration numbers are now larger, but the methods of reflections with our coarse correction are still scalable.

We finally repeat this test with a fixed number of $J = L^2 = 9$ objects of radius $r = 0.1$, but using various mesh sizes for the boundary discretization of the objects ∂O_j and the external boundary $\partial\Omega$. The results are shown in Table 7.3. We observe scalability with respect to the mesh size in all cases.

8. Conclusions

We presented an extensive analysis of the alternating and parallel methods of reflections using domain decomposition techniques. We proved that for two objects, the methods of reflections can be identified with Schwarz domain decomposition methods, which led to new, sharp convergence estimates in this case. We also showed that for more than two objects, the methods of reflections are different from Schwarz domain decomposition methods. We then used substructuring techniques from domain decomposition to rewrite the methods of reflections only iterating on traces, which allowed us to identify these methods with block Gauss-Seidel and block Jacobi methods. Using this insight, we derived new, relaxed variants of the methods of reflections, which converge faster than the classical variants, and also introduced for the first time a coarse correction for the methods of reflections to make them scalable when the number of objects becomes large. The substructured formulation allowed us also naturally to use these methods as preconditioners for Krylov methods. All our results were obtained for the case of the Laplacian, but other operators could be used as well in the relations we found.

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